Канонични параметри на класове повърхнини в 4-мерно пространство на Минковски

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Introduction

Lund-Regge problem:

Find a minimal number of functions, satisfying some natural conditions, that determine the surface up to a motion in a pseudo-Euclidean space.

[Lund F., Regge T., *Unified approach to strings and vortices with soliton solutions*. Phys. Rev. D, 14, no. 6 (1976), 1524–1536]

The problem is solved for:

- Zero mean curvature surfaces of co-dimension two in $\mathbb{E}^4$, $\mathbb{E}^4_1$, and $\mathbb{E}^4_2$;
- Surfaces with parallel normalized mean curvature vector field in $\mathbb{E}^4$, $\mathbb{E}^4_1$, and $\mathbb{E}^4_2$.

Surfaces with zero mean curvature in $\mathbb{E}^4_1$

Alías and Palmer [Math. Proc. Cambridge Philos. Soc., 1998]

Spacelike surfaces with zero mean curvature in $\mathbb{E}^4_1$ are described by the following system of partial differential equations

$$(K^2 + \varkappa^2)^{\frac{1}{4}} \, \Delta \ln(K^2 + \varkappa^2) = 8K$$

$$(K^2 + \varkappa^2)^{\frac{1}{4}} \, \Delta \arctan \frac{\varkappa}{K} = 2\varkappa$$

where K and $\varkappa$ are the Gauss curvature and the normal curvature, respectively.

Conversely, any solution $(K, \varkappa)$ to this system determines a unique (up to a rigid motion in $\mathbb{E}^4_1$) spacelike surface with zero mean curvature whose Gauss curvature and normal curvature are the functions K and $\varkappa$, respectively.

Surfaces with zero mean curvature in $\mathbb{E}^4_1$

G. Ganchev, V.M. [Israel J. Math., 2013]

The Gauss curvature K and the normal curvature $\varkappa$ of any **timelike surface with zero mean curvature** in $\mathbb{E}^4_1$ satisfy the following system of natural partial differential equations

$$(K^2+\varkappa^2)^{\frac{1}{4}}\,\Delta^h \ln(K^2+\varkappa^2) = 8K$$
$$(K^2+\varkappa^2)^{\frac{1}{4}}\,\Delta^h \arctan\frac{\varkappa}{K} = 2\varkappa$$

where Δ^h denotes the hyperbolic Laplace operator $\Delta^h = \frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2}$.
Conversely, any solution $(K,\ \varkappa)$ to the above system, determines a unique (up to a rigid motion in $\mathbb{E}^4_1$) timelike surface with zero mean curvature such that K is the Gauss curvature and $\varkappa$ is the normal curvature of the surface.

Surfaces with zero mean curvature in $\mathbb{E}^4_2$

Sakaki M. [Tsukuba J. Math., 2011]

Spacelike surfaces with zero mean curvature (maximal spacelike surfaces) in $\mathbb{E}^4_2$ are characterized by the following system of partial differential equations:

$$(K^2 - \varkappa^2)^{\frac{1}{4}}\, \Delta \ln(K^2 - \varkappa^2) = 8K$$
$$(K^2 - \varkappa^2)^{\frac{1}{4}}\, \Delta \ln \frac{K - \varkappa}{K + \varkappa} = -4\varkappa \qquad K^2 - \varkappa^2 > 0.$$

The Gauss curvature K and the normal curvature $\varkappa$ of any **maximal spacelike surface** in $\mathbb{E}^4_2$ satisfy the condition

$$K^2 - \varkappa^2 \geq 0.$$

The equality case is the analogue of the super-conformal minimal surfaces in the Euclidean space $\mathbb{E}^4$.

Surfaces with zero mean curvature in $\mathbb{E}^4_2$

Y. Aleksieva, V.M. [J. Geom. Phys., 2019]

The Gauss curvature K and the normal curvature $\varkappa$ (expressed in terms of the canonical isothermal coordinates) of any **minimal Lorentz surface** in $\mathbb{E}^4_2$ satisfy the following system of natural partial differential equations:

$$\begin{aligned} \left|K^2-\varkappa^2\right|^{\frac{1}{4}} \Delta^h \ln \left|K^2-\varkappa^2\right| &= 8K \\ \left|K^2-\varkappa^2\right|^{\frac{1}{4}} \Delta^h \ln \left|\frac{K+\varkappa}{K-\varkappa}\right| &= 4\varkappa \end{aligned} \qquad K^2-\varkappa^2 \neq 0. \qquad (1)$$

Conversely, any solution $(K, \varkappa)$ to this system determines a unique (up to a rigid motion in $\mathbb{E}^4_2$) minimal Lorentz surface of general type with Gauss curvature K and normal curvature $\varkappa$ and such that the given parameters are canonical.

Surfaces with parallel normalized mean curvature vector field

Definition 1

A surface is said to have ***parallel mean curvature vector field*** if its mean curvature vector H is parallel with respect to the normal connection.

Definition 2

A submanifold in a Riemannian manifold is said to have ***parallel normalized mean curvature vector field*** if the mean curvature vector is non-zero and the unit vector in the direction of the mean curvature vector is parallel in the normal bundle [B.-Y. Chen, *Monatsh. Math.*, 1980].

- Every analytic surface with parallel normalized mean curvature vector in the Euclidean m-space $\mathbb{R}^m$ must either lie in a 4-dimensional space $\mathbb{R}^4$ or in a hypersphere of $\mathbb{R}^m$ as a minimal surface [B.-Y. Chen, *Monatshefte für Mathematik* 1980].

Surfaces with parallel normalized mean curvature vector field

G. Ganchev, V.M. [Filomat; 2019]

Each **spacelike surface with parallel normalized mean curvature vector field** in $\mathbb{R}^4_1$ is determined up to a motion by three functions $\lambda(u,v)$, $\mu(u,v)$ and $\nu(u,v)$ satisfying the following system of partial differential equations

$$\nu_u = \lambda_v - \lambda(\ln|\mu|)_v;$$

$$\nu_v = \lambda_u - \lambda(\ln|\mu|)_u;$$

$$\varepsilon(\nu^2 - \lambda^2 + \mu^2) = \frac{1}{2}|\mu|\Delta\ln|\mu|,$$

where $\varepsilon = 1$ corresponds to the case the mean curvature vector field is spacelike, and $\varepsilon = -1$ corresponds to the case the mean curvature vector field is timelike.

Surfaces with parallel normalized mean curvature vector field

Fundamental Theorem 1 [V. Bencheva, V.M., Turkish J. Math., 2024]

Let $\lambda(u,v)$, $\mu(u,v)$ and $\nu(u,v)$ be smooth functions, $\mu \neq 0$, $\nu \neq const$, defined in a domain $\mathcal{D}$, $\mathcal{D} \subset \mathbb{R}^2$, and satisfying the conditions

$$\begin{aligned} &\nu_u + \lambda_v = \lambda(\ln|\mu|)_v; \\ &\lambda_u - \varepsilon\nu_v = \lambda(\ln|\mu|)_u; \\ &|\mu|\,(\ln|\mu|)_{uv} = -\nu^2 - \varepsilon(\lambda^2 + \mu^2), \end{aligned} \qquad (2)$$

where $\varepsilon = \pm 1$. If $\{x_0, y_0, (n_1)_0, (n_2)_0\}$ is a pseudo-orthonormal frame at a point $p_0 \in \mathbb{R}^4_1$, then there exists a subdomain $\mathcal{D}_0 \subset \mathcal{D}$ and a unique timelike surface $\mathcal{M} : z = z(u,v)$, $(u,v) \in \mathcal{D}_0$ with parallel normalized mean curvature vector field, such that $\mathcal{M}$ passes through p_0, $\{x_0, y_0, (n_1)_0, (n_2)_0\}$ is the geometric frame of $\mathcal{M}$ at the point p_0, the functions $\lambda(u,v)$, $\mu(u,v)$, $\nu(u,v)$ are the geometric functions of the surface, and $K - H^2 > 0$ in the case $\varepsilon = 1$, resp. $K - H^2 < 0$ in the case $\varepsilon = -1$. Furthermore, (u,v) are canonical isotropic parameters of $\mathcal{M}$.

Surfaces with parallel normalized mean curvature vector field

Fundamental Theorem 2 [V. Bencheva, V.M., Turkish J. Math., 2024]

Let $\lambda(u,v)$, $\mu(u,v)$ and $\nu(u)$ be smooth functions, $\mu \neq 0$, $\nu \neq const$, defined in a domain $\mathcal{D}$, $\mathcal{D} \subset \mathbb{R}^2$, and satisfying the conditions

$$\begin{aligned} \nu_u + \lambda_v &= \lambda(\ln|\mu|)_v; \\ |\mu|\,(\ln|\mu|)_{uv} &= -\nu^2. \end{aligned} \tag{3}$$

If $\{x_0, y_0, (n_1)_0, (n_2)_0\}$ is a pseudo-orthonormal frame at a point $p_0 \in \mathbb{R}^4_1$, then there exists a subdomain $\mathcal{D}_0 \subset \mathcal{D}$ and a unique timelike surface $\mathcal{M}: z = z(u,v)$, $(u,v) \in \mathcal{D}_0$ with parallel normalized mean curvature vector field, such that $\mathcal{M}$ passes through p_0, $\{x_0, y_0, (n_1)_0, (n_2)_0\}$ is the geometric frame of $\mathcal{M}$ at the point p_0, the functions $\lambda(u,v)$, $\mu(u,v)$, $\nu(u)$ are the geometric functions of the surface, and $K - H^2 = 0$. Furthermore, (u,v) are canonical isotropic parameters of $\mathcal{M}$.

Question

Question 1

How to introduce canonical parameters and obtain natural equations for other classes of surfaces, different from the minimal ones and from the PNMCVF-surfaces?

Question 2

Can we solve the Lund-Regge problem for other classes of surfaces, different from the minimal ones and from the PNMCVF-surfaces?

We solve this problem for the class of **marginally trapped surfaces** in the Minkowski 4-space $\mathbb{E}^4_1$.

Marginally trapped surfaces in the Minkowski 4-space

Theorem 1 [G. Ganchev, V.M., *J. Math. Phys.* 2012]

Let $\gamma_1, \gamma_2, \nu, \lambda, \mu, \beta_1, \beta_2$ *be smooth functions, defined in a domain* $\mathcal{D}, \ \mathcal{D} \subset \mathbb{R}^2$*, and satisfying the conditions*

$$\frac{\mu_u}{\mu(2\,\gamma_2 + \beta_1)} > 0; \qquad \frac{\mu_v}{\mu(2\,\gamma_1 + \beta_2)} > 0;$$

$$-\gamma_1 \sqrt{E}\sqrt{G} = (\sqrt{E})_v; \qquad -\gamma_2 \sqrt{E}\sqrt{G} = (\sqrt{G})_u;$$

$$2\lambda\,\mu = \frac{1}{\sqrt{E}}\,(\gamma_2)_u + \frac{1}{\sqrt{G}}\,(\gamma_1)_v - \left((\gamma_1)^2 + (\gamma_2)^2\right);$$

$$2\lambda\,\gamma_2 - 2\nu\,\gamma_1 - \lambda\,\beta_1 + (1+\nu)\,\beta_2 = \frac{1}{\sqrt{E}}\,\lambda_u - \frac{1}{\sqrt{G}}\,\nu_v;$$

$$2\lambda\,\gamma_1 + 2\nu\,\gamma_2 + (1-\nu)\,\beta_1 - \lambda\,\beta_2 = \frac{1}{\sqrt{E}}\,\nu_u + \frac{1}{\sqrt{G}}\,\lambda_v;$$

$$\gamma_1\,\beta_1 - \gamma_2\,\beta_2 + 2\nu\,\mu = -\frac{1}{\sqrt{E}}\,(\beta_2)_u + \frac{1}{\sqrt{G}}\,(\beta_1)_v,$$

where $\sqrt{E} = \dfrac{\mu_u}{\mu(2\,\gamma_2 + \beta_1)}$, $\sqrt{G} = \dfrac{\mu_v}{\mu(2\,\gamma_1 + \beta_2)}$.
Let $\{x_0,\, y_0,\, (n_1)_0,\, (n_2)_0\}$ *be vectors at a point* $p_0 \in \mathbb{R}^4_1$, *such that* x_0, y_0 *are unit spacelike vectors,* $\langle x_0, y_0\rangle = 0$, $(n_1)_0, (n_2)_0$ *are lightlike vectors, and* $\langle (n_1)_0, (n_2)_0\rangle = -1$. *Then there exist a subdomain* $\mathcal{D}_0 \subset \mathcal{D}$ *and a unique marginally trapped surface* $M^2 : z = z(u, v),\ (u, v) \in \mathcal{D}_0$ *free of flat points, such that* M^2 *passes through* p_0, *the functions* $\gamma_1,\, \gamma_2,\, \nu,\, \lambda,\, \mu,\, \beta_1, \beta_2$ *are the geometric functions of* M^2 *and* $\{x_0,\, y_0,\, (n_1)_0,\, (n_2)_0\}$ *is the geometric frame of* M^2 *at the point* p_0.

Marginally trapped surfaces in the Minkowski 4-space

Theorem 2 [M. Maksimovic, V.M., 2025]

Let $\nu(u,v)$, $\lambda(u,v)$, *and* $\mu(u,v)$ $(\mu \neq 0)$ *be smooth functions defined in a domain* $\mathcal{D} \subset \mathbb{R}^2$ *and* $\phi_i(u,v)$, $i = 1,2,3,4$ *be defined by* (5)*. Let* $\varphi(u,v)$, $\psi(u,v)$ *be a solution to the Cauchy problem*

$$\begin{aligned} \varphi_v &= \phi_1\varphi + \phi_2\psi; \\ \psi_u &= \phi_3\varphi + \phi_4\psi; \end{aligned} \qquad \varphi(u,v_0) = g_1(u); \;\; \psi(u_0,v) = g_2(v),$$

where $g_1(u)$ *and* $g_2(v)$ *are defined by* (6)*, and let the following equations also hold*

$$\begin{aligned} 2\lambda\mu &= -\frac{1}{\varphi\psi}\left(\left(\frac{\varphi_v}{\psi}\right)_v + \left(\frac{\psi_u}{\varphi}\right)_u\right); \\ 2\nu\mu &= \frac{2}{\varphi\psi}\left(\left(\frac{\psi_u}{\psi}\right)_v - \left(\frac{\varphi_v}{\varphi}\right)_u\right). \end{aligned} \qquad (4)$$

Then, there exists a unique (up to a position in $\mathbb{R}^4_1$*) marginally trapped surface free of flat points parametrized by canonical principal parameters* (u,v) *with geometric functions* $\nu(u,v)$, $\lambda(u,v)$, *and* $\mu(u,v)$.

$$\begin{aligned}
\phi_1 &= -\frac{\mu(\lambda^2+\nu^2-\nu)_v + (2(\lambda^2+\nu^2)+\nu-1)\mu_v}{2\mu(4\nu^2+4\lambda^2-1)} \ ; \\
\phi_2 &= \frac{2\mu(\lambda_u\nu-\lambda\nu_u)+\lambda\mu_u-\lambda_u\mu}{2\mu(4\nu^2+4\lambda^2-1)} \ ; \\
\phi_3 &= \frac{2\mu(\lambda\nu_v-\lambda_v\nu)+\lambda\mu_v-\lambda_v\mu}{2\mu(4\nu^2+4\lambda^2-1)} \ ; \\
\phi_4 &= -\frac{\mu(\lambda^2+\nu^2+\nu)_u + (2(\lambda^2+\nu^2)-\nu-1)\mu_u}{2\mu(4\nu^2+4\lambda^2-1)} \ .
\end{aligned} \tag{5}$$

$$\begin{aligned}
g_1(u) &= e^{\int_{u_0}^{u} (c\phi_3+\phi_4)(u,v_0)du - c_1} , \\
g_2(v) &= e^{\int_{v_0}^{v} \left(\phi_1+\frac{1}{c}\phi_2\right)(u_0,v)dv - c_2} .
\end{aligned} \tag{6}$$

Lemma

If (u, v) and $(\bar{u}, \bar{v})$ are two pairs of canonical principal parameters in a neighbourhood of a point p, then the following relations hold

$$\bar{u} = \pm u + u_0; \qquad \bar{v} = \pm v + v_0,$$

or

$$\bar{u} = \pm v + v_0; \qquad \bar{v} = \pm u + u_0 \ .$$

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Thank you for your attention!