Existence of solitary wave solution for the Benjamin equation

December 16th, 2025

- We consider the Benjamin equation

$$u_t - u_{xxx} - \gamma D u_x + 2uu_x = 0 \tag{1}$$

- where $D = H\partial_x$, where H is the Hilbert transform.
- The Hilbert transform is defined via the convolution with the distribution $\frac{1}{x}$, i.e. $Hf = \frac{1}{x} * f$ or equivalently

$$Hf(x) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{f(x-y)}{y} dy.$$

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- let $u(t, x) = \phi(x + ct)$, with ϕ vanishing at both $\pm\infty$.
- the profile equation

$$-\phi'' - \gamma D\phi + c\phi + \phi^2 = 0. \qquad (2)$$

- A rescaling transformation like $\phi \to -\frac{\gamma}{2}\phi(\frac{\gamma}{2}\cdot)$ will transform the problem into an equivalent one

$$-\phi'' - 2D\phi + 4c\gamma^{-2}\phi - \phi^2 = 0. \qquad (3)$$

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- one should expect solutions to exists only if the dispersion relation, $\xi^2 - 2|\xi| + 4c\gamma^{-2} > 0$ for all values of ξ. This implies that $4c\gamma^{-2} > 1$ or $c > \frac{\gamma^2}{4}$.
  we consider an equivalent version of the problem (3)

$$-\phi'' - 2D\phi + (\omega + 1)\phi - \phi^2 = 0. \tag{4}$$

Note that due to the fact $D^2 = H^2\partial_x^2 = -\partial_x^2$, we can equivalently rewrite in the more convenient form

$$(D - 1)^2\phi + \omega\phi - \phi^2 = 0. \tag{5}$$

where $\omega = 4c\gamma^{-2} - 1$

- We start with a simple interpolation inequality.

Lemma

Let $p = 3, 4, 5, 6$. Then, for every $\alpha > 0$, there exists $C_{\alpha,p}$, so that

$$\int_{-\infty}^{\infty} u^p(x)dx \le C_{\alpha,p}\|u\|^{p-2}\left(\|(D-1)u\|^2 + \alpha\|u\|^2\right). \tag{6}$$

Moreover, for $2 < p \le 6$ and any $\alpha > 0$, there exists $D_{\alpha,p}$, so that

$$\int_{-\infty}^{\infty} |u(x)|^p dx \le D_{\alpha,p}\|u\|^{p-2}\left(\|(D-1)u\|^2 + \alpha\|u\|^2\right). \tag{7}$$

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- Compensated compactness

Theorem

Let $\rho_n : \mathbf{R} \to \mathbf{R}$, $\rho_n \geq 0$, $\int_{-\infty}^{\infty} \rho_n(x)dx = \lambda$. Then, there exists a subsequence ρ_{n_k}, so that one of the following is true:
(Compactness/tightness) there exists a sequence $y_k \in \mathbf{R}$, so that for every $\epsilon > 0$, there exists $R > 0$, and k_0, so that for all $k \geq k_0$,

$$\int_{y_k-R}^{y_k+R} \rho_{n_k}(x)dx \geq \lambda - \epsilon.$$

(Vanishing) For all $R < \infty$,

$$\lim_{k\to\infty} \sup_{y\in\mathbf{R}} \int_{y-R}^{y+R} \rho_{n_k}(x)dx = 0.$$

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- (Dichotomy) There exists $\mu \in (0, \lambda)$, so that for all $\epsilon > 0$, there exists $y_k \in \mathbf{R}$ and $\rho_{k,+}, \rho_{k,-} \in L^1_+(\mathbf{R})$, so that for all k large enough, $\rho_{n_k}(x + y_k) = \rho_{k,+}(x) + \rho_{k,-}(x) + e_k$ and

  $$supp(\rho_{k,-}) \subset (-\infty, -R_k), supp(\rho_{k,+}) \subset (R_k, +\infty),$$
  $$\lim_k R_k = \infty$$
  $$|\int_{R_k}^{+\infty} \rho_{k,+}(x)dx - \mu| < \epsilon, |\int_{-\infty}^{-R_k} \rho_{k,-}(x)dx - (\lambda - \mu)| < \epsilon,$$
  $$\int |e_k(x)|dx < \epsilon.$$

We consider the inequality (6) for the case $p = 3$ and arbitrary $\alpha > 0$,

$$\int_{-\infty}^{\infty} u^3(x)dx \leq C_\alpha \|u\| \left(\|(D-1)u\|^2 + \alpha \|u\|^2 \right). \qquad (8)$$

Here we take the value C_α to be the exact constant in (8). In other words,

$$C_\alpha = \sup_{u \neq 0} I_\alpha[u]$$

$$I_\alpha[u] = \frac{\int_{-\infty}^{\infty} u^3(x)dx}{\|u\| \left(\|(D-1)u\|^2 + \alpha \|u\|^2 \right)}.$$

It is obviously not at all clear that a maximizer exists. This is the subject of the following proposition.
For each $\alpha > 0$, there exists a maximizer for (8). That is, there exists a function $\varphi \in H^1(\mathbf{R})$, so that

$$C_\alpha = \frac{\int_{-\infty}^{\infty} \varphi^3(x)dx}{\|\varphi\| \left(\|(D-1)\varphi\|^2 + \alpha\|\varphi\|^2\right)}.$$

Thank you for Attention