Estimates for a class of functions that generalize the Mittag-Leffler function

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Simplest fractional differential equation

${}^{C}D_t^{\alpha}$ - Caputo fractional derivative:

$$ {}^{C}D_t^{\alpha} = J_t^{n-\alpha}\frac{\mathsf{d}^n}{\mathsf{d}t^n}, \qquad n-1<\alpha\le n,\ n\in\mathbb{N}, $$

where J_t^{β} - Riemann-Liouville fractional integral

$$ \left(J_t^{\beta} f\right)(t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1} f(\tau)\,\mathsf{d}\tau, \quad \beta>0; \qquad J_t^0 = I. $$

Fractional relaxation-oscillation equation

$$ {}^{C}D_t^{\alpha}u(t) + \lambda u(t) = 0, \quad 0<\alpha\le 2,\ \lambda>0,\ t>0; $$
$$ u(0) = 1 \ (\text{and } u'(0)=0 \text{ if } \alpha>1). $$

Solution: $u(t) = E_{\alpha}(-\lambda t^{\alpha})$, where

$$ E_{\alpha}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)} $$ – **Mittag-Leffler function**

Two-parameter Mittag-Leffler function

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \quad E_\alpha(z) = E_{\alpha,1}(z)$$

Asymptotic expansion:

$$E_{\alpha,\beta}(-t) = \frac{t^{-1}}{\Gamma(\beta-\alpha)} - \frac{t^{-2}}{\Gamma(\beta-2\alpha)} + O(t^{-3}), \quad \alpha \in (0,2), \beta \in \mathbb{R}, t \to +\infty.$$

Estimate:

$$|E_{\alpha,\beta}(-t)| \leq \frac{C}{1+t}, \quad t \geq 0.$$

Complete monotonicity:

A function $f : (0,\infty) \to \mathbb{R}$ is called **completely monotone function** ($\mathcal{CMF}$) if

$$(-1)^n f^{(n)}(t) \geq 0, \text{ for all } t > 0, \; n = 0, 1, \ldots$$

The simplest example: $E_1(-t) = e^{-t} \in \mathcal{CMF}$
$E_\alpha(-t) \in \mathcal{CMF}$ iff $0 < \alpha \leq 1$ (Pollard, 1948)
$E_{\alpha,\beta}(-t) \in \mathcal{CMF}$ iff $0 \leq \alpha \leq 1$, $\alpha \leq \beta$.

Multinomial Mittag-Leffler function

$$E_{(\mu_1,\ldots,\mu_m),\beta}(z_1,\ldots,z_m) := \sum_{k=0}^{\infty} \sum_{k_1+\cdots+k_m=k} \frac{k!}{k_1!\cdots k_m!} \frac{\prod_{j=1}^m z_j^{k_j}}{\Gamma\left(\beta+\sum_{j=1}^m \alpha_j k_j\right)},$$

where $z_j \in \mathbb{C}, \mu_j > 0, \beta \in \mathbb{R}, j = 1, ..., m.$

Multinomial Mittag-Leffler type functions in the form

$$\mathcal{E}_{(\mu_1,\ldots,\mu_m),\beta}(t; a_1,\ldots,a_m) := t^{\beta-1} E_{(\mu_1,\ldots,\mu_m),\beta}(-a_1 t^{\mu_1},\ldots,-a_m t^{\mu_m}).$$

appear in the solution of differential equations with multiple fractional derivatives. This is due to the Laplace transform identity:

$$\int_0^t e^{-st} \mathcal{E}_{(\mu_1,\ldots,\mu_m),\beta}(t; a_1,\ldots,a_m)\,\mathrm{d}t = \frac{s^{-\beta}}{1+\sum_{j=1}^m a_j s^{-\mu_j}}$$

Multi-term fractional relaxation equation

$$ {}^{C}D_t^{\alpha} y(t) + \sum_{j=1}^{m} b_j \, {}^{C}D_t^{\alpha_j} y(t) + \lambda y(t) = f(t), \qquad t > 0, \quad y(0) = 1, $$

where $1 \geq \alpha > \alpha_j > 0$, $b_j > 0$, $j = 1, ..., m$, $\lambda > 0$.

Solution:

$$ y(t) = G_1(t) + \int_0^t G(t-\tau) f(\tau) \, d\tau, $$

where

$$ \begin{aligned} G_1(t) &= 1 - \lambda \mathcal{E}_{(\alpha, \alpha-\alpha_1, \ldots, \alpha-\alpha_m), \alpha+1} \left(t; \lambda, b_1, \ldots, b_m \right), \\ G(t) &= \mathcal{E}_{(\alpha, \alpha-\alpha_1, \ldots, \alpha-\alpha_m), \alpha} \left(t; \lambda, b_1, \ldots, b_m \right). \end{aligned} $$

The functions $G_1(t), G(t)$ are completely monotone as in the single-term case.

Generalizations of Prabhakar type

Three-parameter Mittag-Leffler function (Prabhakar function):

$$E^{\delta}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\delta)_k}{k!} \frac{z^k}{\Gamma(\beta+\alpha k)}, \qquad z \in \mathbb{C},\ \alpha,\beta,\delta \in \mathbb{R},\ \alpha > 0, \tag{1}$$

where $(\delta)_k$ is the Pochhammer symbol

$$(\delta)_k = \frac{\Gamma(\delta+k)}{\Gamma(\delta)} = \delta(\delta+1)\ldots(\delta+k-1), \quad k \in \mathbb{N}; \qquad (\delta)_0 = 1.$$

The function $t^{\beta-1}E^{\delta}_{\alpha,\beta}(-t^{\alpha}),\ t \geq 0$, is **completely monotone** provided

$$0 < \alpha \leq 1,\ 0 < \alpha\delta \leq \beta \leq 1. \tag{2}$$

Asymptotic behavior: $E^{\delta}_{\alpha,\beta}(-t^{\alpha}) \sim \begin{cases} \dfrac{t^{-\alpha\delta}}{\Gamma(\beta-\alpha\delta)}, & \alpha\delta \neq \beta, \\ -\dfrac{\delta t^{-\alpha\delta-\alpha}}{\Gamma(-\alpha)}, & \alpha\delta = \beta. \end{cases}, \quad t \to +\infty,$

Multinomial generalization of Prabhakar type

$$E^{\delta}_{(\mu_1,\ldots,\mu_m),\beta}(z_1,\ldots,z_m) := \sum_{k=0}^{\infty} \sum_{k_1+\cdots+k_m=k} \frac{(\delta)_k}{k_1!\cdots k_m!} \frac{\prod_{j=1}^{m} z_j^{k_j}}{\Gamma\left(\beta+\sum_{j=1}^{m}\mu_j k_j\right)},$$

where where $z_j \in \mathbb{C}$, $\mu_j, \beta, \delta \in \mathbb{R}$, $\mu_j > 0$, $j = 1, \ldots, m$.

Naturally emerging function in the study of IBVPs for multi-term time-fractional equations with **non-local boundary conditions**.

Spectral expansion of the solution in case of two-dimensional eigenspaces:

$$u(x,t) = u_{1,0}(t)\Phi_{1,0}(x) + \sum_{n=1}^{\infty} \{u_{1,n}(t)\varphi_{1,n}(x) + u_{2,n}(t)\varphi_{2,n}(x)\}$$

$\varphi_{1,n}(x)$ - eigenfunctions, $\varphi_{2,n}(x)$ - associated eigenfunctions (due to non-locality)
$u_{1,n}(t)$ - solutions of relaxation equation, represented in terms of multinomial Mittag-Leffler functions
$u_{2,n}(t)$ contains convolutions of multinomial Mittag-Leffler type functions.

Multinomial Mittag-Leffler type function of Prabhakar type

Class of functions, closed under convolution:

$$\mathcal{E}^{\delta}_{\vec{\mu},\beta}(t;\vec{a}) := t^{\beta-1} E^{\delta}_{(\mu_1,\ldots,\mu_m),\beta}(-a_1 t^{\mu_1},\ldots,-a_m t^{\mu_m})$$

Convolution property: $\int_0^t \mathcal{E}^{\delta}_{\vec{\mu},\beta}(t-\tau;\vec{a})\mathcal{E}^{\delta_0}_{\vec{\mu},\beta_0}(\tau;\vec{a})\,\mathrm{d}t = \mathcal{E}^{\delta+\delta_0}_{\vec{\mu},\beta+\beta_0}(t;\vec{a})$.

Laplace transform:

$$\int_0^t e^{-st}\mathcal{E}^{\delta}_{\vec{\mu},\beta}(t;\vec{a})\,\mathrm{d}t = \frac{s^{-\beta}}{\left(1+\sum_{j=1}^m a_j s^{-\mu_j}\right)^{\delta}}$$

Complete monotonicity: Let $\mu_j, a_j > 0$, $j = 1,\ldots,m$, and

$$\mu_1 \le 1,\ 0 < \mu_1\delta \le \beta \le 1,$$

where $\mu_1 = \max\{\mu_j\}_{j=1}^m$. Then

$$\mathcal{E}^{\delta}_{\vec{\mu},\beta}(t;\vec{a}) \in \mathcal{CMF},\quad t > 0.$$

A nonlocal BVP for the general time-fractional diffusion equation

$$
\begin{aligned}
&{}^{C}\mathbb{D}_t^{(\zeta)} u(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t), \quad x \in (0,1),\ t > 0,\\
&u(x,0) = f(x),\\
&u(0,t) = 0, \quad \Phi_x\{u(x,t)\} = 0,
\end{aligned}
$$

where ${}^{C}\mathbb{D}_t^{(\zeta)}$ is a general fractional derivative in time with kernel $\zeta(t)$ and Φ_x is a continuous linear functional in $C^1[0,1]$.

P_{λ_n} - spectral projection operators in the (multidimensional) eigenspaces, defined by the spectral problem

$$
y'' + \lambda^2 y = 0,\ y(0) = 0,\ \Phi\{y\} = 0.
$$

The spectral expansion

$$
f(x) \sim \sum_{n=0}^{\infty} P_{\lambda_n} f
$$

has the uniqueness property iff $1 \in \operatorname{supp}\, \Phi$ (N. Bozhinov, 1990).

Aim: to construct solution $u(x,t)$ in the form of spectral expansion

$$u(x,t)=\sum_{n=0}^{\infty} P_{\lambda_n}u, \quad P_{\lambda_n}u=\sum_{k=0}^{\kappa_n-1} A_{n,k}(t)\varphi_{n,k}(x) \tag{3}$$

where κ_n - multiplicity of eigenvalue λ_n, $\varphi_{n,k}(x)$ - generalized eigenfunctions, $A_{n,k}(t)$ - represented in terms of functions $G(t)$ and $G_j(t)$, $j=1,...,\kappa_n-1$, defined as

$$\widehat{G}_1(s;\lambda)=\frac{g(s)}{s(g(s)+\lambda)}, \quad \widehat{G}(s;\lambda)=\frac{1}{g(s)+\lambda}, \quad g(s)=s\widehat{\zeta}(s),$$

$$G_{j+1}(t;\lambda)=\int_0^t G(\tau;\lambda)G_j(t-\tau;\lambda)\,d\tau, \quad j\in\mathbb{N}.$$

Example: If ${}^C\mathbb{D}_t^{(\zeta)}={}^CD_t^{\alpha}$ then

$$G_1(t;\lambda)=E_\alpha(-\lambda t^\alpha), \quad G(t;\lambda)=t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha),$$

$$G_{j+1}(t;\lambda)=t^{\alpha j}E_{\alpha,\alpha j+1}^{j+1}(-\lambda t^\alpha), \; j\in\mathbb{N}.$$

Example: If ${}^C\mathbb{D}_t^{(\zeta)} = {}^C D_t^{\alpha} + \sum_{j=1}^m b_j \, {}^C D_t^{\alpha_j}$ then

$$\begin{aligned} G_1(t;\lambda) &= 1 - \lambda \mathcal{E}_{(\alpha,\alpha-\alpha_1,\ldots,\alpha-\alpha_m),\alpha+1}\left(t;\lambda,b_1,\ldots,b_m\right), \\ G(t;\lambda) &= \mathcal{E}_{(\alpha,\alpha-\alpha_1,\ldots,\alpha-\alpha_m),\alpha}\left(t;\lambda,b_1,\ldots,b_m\right), \\ G_{j+1}(t;\lambda) &= \mathcal{E}^{j}_{(\alpha,\alpha-\alpha_1,\ldots,\alpha-\alpha_m),\alpha j+1}\left(t;\lambda,b_1,\ldots,b_m\right) \\ &- \lambda \mathcal{E}^{j+1}_{(\alpha,\alpha-\alpha_1,\ldots,\alpha-\alpha_m),\alpha(j+1)+1}\left(t;\lambda,b_1,\ldots,b_m\right), \ j \in \mathbb{N}. \end{aligned}$$

To prove convergence of the series (3), estimates for $A_{n,k}(t)$ (resp. $G_j(t;\lambda)$) are derived in the general case.

Theorem. For any $\lambda > 0$ the functions $G_j(t;\lambda)$, $j \in \mathbb{N}$, are positive and continuous in $t \in [0,\infty)$ and satisfy the following estimates:

$$\lambda^{j-1} G_j(t;\lambda) \leq 1, \quad t \geq 0, \tag{4}$$

$$\lambda^{j} G_j(t;\lambda) \leq C_\varepsilon, \quad t \geq \varepsilon > 0, \tag{5}$$

where the constant $C_\varepsilon > 0$ does not depend on t or λ.

Publications:

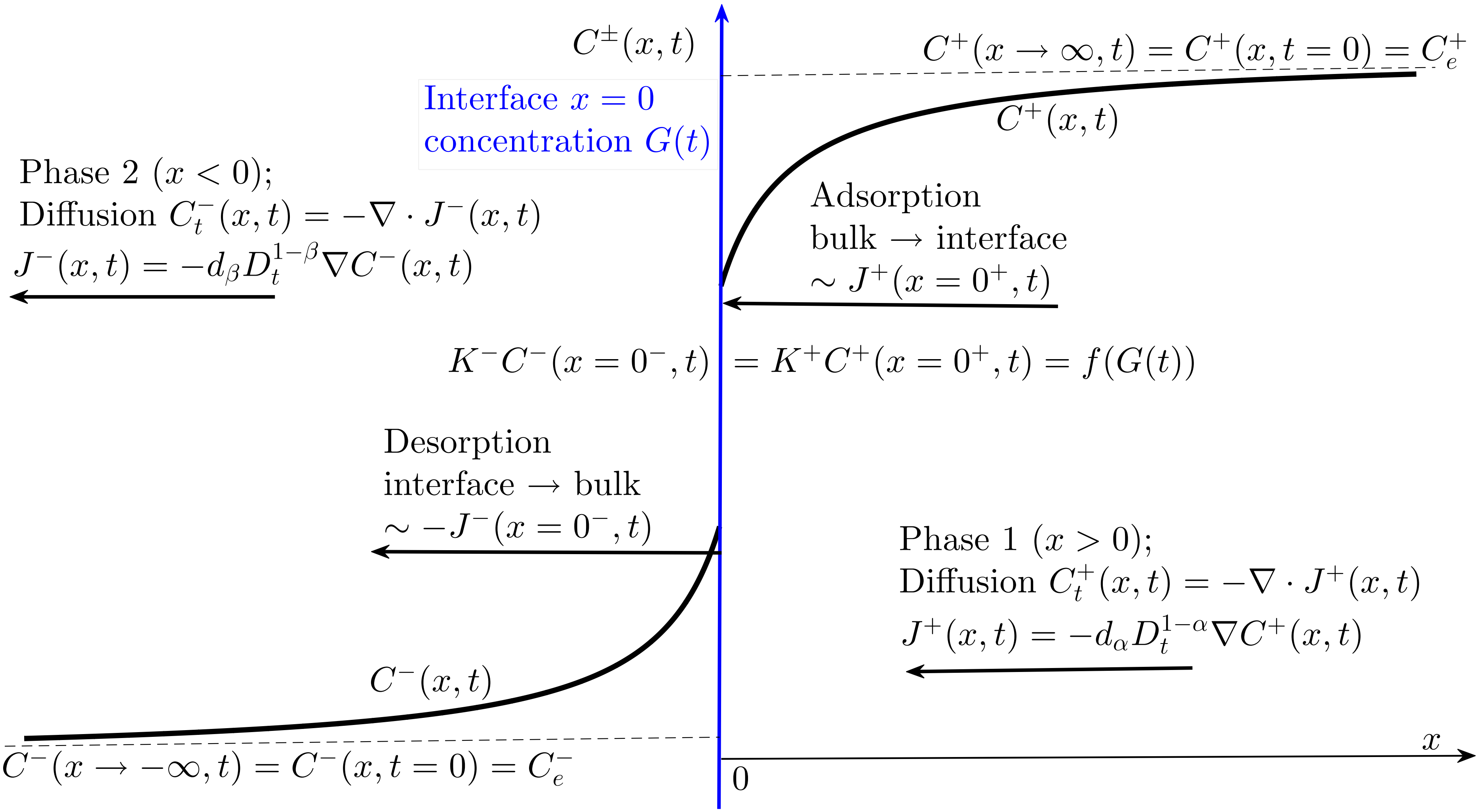
E. Bazhlekova (2025) Regarding a Class of Nonlocal BVPs for the General Time-Fractional Diffusion Equation, *Fractal Fract.*, 9 (9), 613.

E. Bazhlekova, Yulian Tsankov (2025) Explicit Solution of a BVP for the Time-Fractional Diffusion Equation with Triple Eigenvalues, submitted to *Fractional Differential Calculus*.

Presentation:

E. Bazhlekova, Multinomial Mittag-Leffler type functions: basic properties and some applications, Fractional Calculus Seminars, SISSA, International School of Advanced Studies, Trieste, Italy, May 9, 2025.

Application: adsorption-desorption at anomalous diffusion



Two time-fractional equations for anomalous diffusion of surfactant:

$$
{}^{C}D_t^{\alpha} C^{+}(x,t) = d_{\alpha} \frac{\partial^2 C^{+}(x,t)}{\partial x^2},\; x>0; \quad {}^{C}D_t^{\beta} C^{-}(x,t) = d_{\beta} \frac{\partial^2 C^{-}(x,t)}{\partial x^2},\; x<0,
$$

where $0<\alpha,\beta \le 1$, d_α, d_β - diffusion coefficients. Initial and boundary conditions:

$$
\begin{gathered}
C^{+}(x,0) = C_e^{+}, \quad x>0; \quad C^{-}(x,0) = C_e^{-}, \quad x<0; \\
\lim_{x\to\pm\infty} C^{\pm}(x,t) = C_e^{\pm}, \quad \lim_{x\to 0^{\pm}} C^{\pm}(x,t) = C_s^{\pm}(t), \quad t>0,
\end{gathered}
$$

where $C_s^{\pm}(t)$ - subsurface surfactant concentrations, $C_e^{\pm}$ - input concentrations of surfactant in the bulk phases.
$G(t)$ - surfactant concentration on the interface:

$$
\frac{dG(t)}{dt} = d_\alpha D_t^{1-\alpha} \left.\frac{\partial C^{+}(x,t)}{\partial x}\right|_{x=0^{+}} - d_\beta D_t^{1-\beta} \left.\frac{\partial C^{-}(x,t)}{\partial x}\right|_{x=0^{-}},\; t>0;\; G(0)=G_0.
$$

ODE for $G(t)$:

$$
\frac{dG(t)}{dt} + d_\alpha D_t^{1-\alpha/2} C_s^{+}(t) + d_\beta D_t^{1-\beta/2} C_s^{-}(t) = d_\alpha C_e^{+} \frac{t^{\alpha/2-1}}{\Gamma(\alpha/2)} + d_\beta C_e^{-} \frac{t^{\beta/2-1}}{\Gamma(\beta/2)}
$$

The subsurface surfactant concentrations $C_s^{\pm}(t)$ are related with $G(t)$ via the adsorption isotherm $f(G)$: $K^- C_s^-(t) = K^+ C_s^+(t) = f(G(t))$

In the simplest case $f(G) = const.G$ the equation for $G(t)$ is

$$\frac{dG(t)}{dt} + aD_t^{1-\alpha/2} G(t) + bD_t^{1-\beta/2} G(t) = a_1 \frac{t^{\alpha/2-1}}{\Gamma(\alpha/2)} + b_1 \frac{t^{\beta/2-1}}{\Gamma(\beta/2)}, \quad G(0) = G_0$$

$$\begin{aligned} \Longrightarrow G(t) \quad &= \quad G_0 E_{(\alpha/2,\beta/2),1}\left(-at^{\alpha/2}, -bt^{\beta/2}\right) \\ &+ \quad a_1 t^{\alpha/2} E_{(\alpha/2,\beta/2),\alpha/2+1}\left(-at^{\alpha/2}, -bt^{\beta/2}\right) \\ &+ \quad b_1 t^{\beta/2} E_{(\alpha/2,\beta/2),\beta/2+1}\left(-at^{\alpha/2}, -bt^{\beta/2}\right). \end{aligned}$$

This solution serves as an approximation for the nonlinear cases.

I. Bazhlekov, E. Bazhlekova (2025) Adsorption-desorption at anomalous diffusion: Fractional calculus approach, $Fractal\ Fract.$, 9 (7), 408.

I. Bazhlekov, E. Bazhlekova (2025) Fractional calculus approach to models of adsorption: Barrier-diffusion control, arXiv preprint.