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Heterotic Supersymmetry, Strominger System and Equations of Motion

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In all these cases the Ricci tensor vanishes authomatically, $Ric^g = 0$ and solves the Einstein equations of motion.

The Einstein equations of motion

$$Ric^g = 0 \tag{1}$$

are highly non-linear PDE's and constructing compact solutions is extremelly difficult.

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One possible theory generalizing the Einstein general relativity incorporating additional forces is the so called heterotic string theory.

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- The bosonic fields are non-trivial only on M^d, $d \leq 8$.
- The two torsion connections $\nabla^{\pm} = \nabla^g \pm \frac{1}{2}H$, ∇^g is the Levi-Civita connection of the Riemannian metric g.
- Both connections preserve the metric, $\nabla^{\pm}g = 0$ and have totally skew-symmetric torsion $\pm H$, respectively.
- An unknown connection ∇ on the tangent bundle.
- R^g, $R^{\pm}$, R - the corresponding curvatures.

The action and equations of motion

The bosonic part of the ten-dimensional supergravity action in the string frame is

$$S = \int e^{-2\phi}\left[Scal^g + 4(\nabla^g \phi)^2 - \frac{1}{2}|H|^2 - \frac{\alpha'}{4}\Big(Tr|F^A|^2) - Tr|R|^2\Big)\right] vol.$$

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- The string frame field equations (the equations of motion) are;

$$\begin{aligned} &Ric^g_{ij} - \frac{1}{4}H_{imn}H_j^{mn} + 2\nabla^g_i\nabla^g_j\phi - \frac{\alpha'}{4}\left[(F^A)_{imab}(F^A)_j^{mab} - R_{imnq}R_j^{mnq}\right] = 0, \\ &\nabla^g_i(e^{-2\phi}H^i_{jk}) = 0, \qquad \nabla^+_i(e^{-2\phi}(F^A)^i_j) = 0. \end{aligned} \tag{2}$$

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- The Green-Schwarz anomaly cancellation mechanism requires that the three-form Bianchi identity receives an α' correction of the form

$$dH = \frac{\alpha'}{4}8\pi^2(p_1(M^d) - p_1(E)) = \frac{\alpha'}{4}\Big(Tr(R\wedge R) - Tr(F^A\wedge F^A)\Big), \tag{3}$$

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This is a kind of a generalization of the Einstein-Hilbert gravity action in the vacuum

$$S = \int \Big[Scal^g\Big] vol \quad and \quad equations \quad of \quad motion \quad Ric = 0.$$

Heterotic supersymmetry and the Strominger system

A heterotic geometry preserves supersymmetry iff there exists at least one Majorana-Weyl spinor ϵ such that the following Killing-spinor equations hold

- $$\begin{aligned} \delta_\lambda &= \nabla_m \epsilon = \left(\nabla^g_m + \frac{1}{4} H_{mnp}\Gamma^{np}\right)\epsilon = \nabla^+\epsilon = 0, \\ \delta_\Psi &= \left(\Gamma^m \partial_m \phi - \frac{1}{12} H_{mnp}\Gamma^{mnp}\right)\epsilon = (d\phi - \frac{1}{2}H)\cdot\epsilon = 0, \\ \delta_\xi &= F^A_{mn}\Gamma^{mn}\epsilon = F^A \cdot \epsilon = 0, \end{aligned} \tag{4}$$

- λ, Ψ, ξ are the gravitino, the dilatino and the gaugino fields,
- Γ_i generate the Clifford algebra $\{\Gamma_i, \Gamma_j\} = 2g_{ij}$
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- The last equation in (4) is the instanton condition which means that the curvature F^A is contained in a Lie algebra of a Lie group which is a stabilizer of a non-trivial spinor.

Heterotic supersymmetry and equations of motion

Basic Problem: In the presence of a curvature term $Tr(R \wedge R)$ the solutions of the Strominger system (4), (3) obey the second and the third equations of motion (the second and the third equations in (2)) but do not always satisfy the Einstein equations of motion.

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Theorem (Iv., Phys. Lett. B - 2010)

The solutions of the Strominger system ((4) and (3)) also solve the heterotic supersymmetric equations of motion (2) if and only if the curvature R of the connection on the tangent bundle is an instanton in dimensions 5,6,7,8.

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- This theorem connects the algebraic geometry notion of stability with differential geometry and non-linear PDE notion of instantons due to the Hitchin- Kobayashi correspondence.

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Studying the moduli space of $SU(2)$-instantons in dimension 4 S.Donaldson establishes the break-down result for the existence of non-diffeomorphic smooth structures on R^4.

Some open questions

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1. Classify, if possible) hermitian manifolds with STABLE TANGENT bundle of degree zero
2. Is it possible to define the notion of stabilty for G_2 and $Spin(7)$ manifolds?
3. Investigate compact G_2 and $Spin(7)$ manifolds admitting a G_2 and $Spin(7)$ instanton, respectively.

In the G_2 case we have

Theorem (Iv-Stanchev, 24)

Let (M,ϕ) be an integrable G_2 manifold of constant type and the curvature of the characteristic connection ∇ is a Ricci flat G_2-instanton, i.e.

$$d*\phi=\theta\wedge *\phi \qquad (d\phi,*\phi)=const., \quad R(X,Y)\in g_2, \quad Ric=0.$$

Then the torsion 3-form is harmonic, $\delta T=dT=0$.
Moreover, the covariant derivatives of the 3-form T with respect to the Levi-Civita connection and the characteristic connection coincide, $\nabla^g T=\nabla T$.

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As a consequence of Theorem 3, we obtain

Theorem (Iv-Stanchev 24)

On an integrable G_2 manifold of constant type, the next conditions are equivalent:

a) *The characteristic connection has curvature $R \in S^2\Lambda^2$ with vanishing Ricci tensor;*
b) *The curvature of the characteristic connection satisfies the Riemannian first Bianchi identity.*
c) *The torsion 3 form is parallel with respect to the Levi-Civita and to the characteristic connections simultaneously, $\nabla^g T = \nabla T = 0$.*

In these cases the exterior derivative $d\phi$ of the G_2-form ϕ is ∇-parallel, $\nabla(d\phi) = 0$.

The physically relevant connection on the tangent bundle to be considered in (3), (2) is the $(-)$-connection, Bergshoeff, de Roo' 89, Hull' 86.
- Reason: the curvature R^- is an instanton up to the first order of α'.
- a consequence of the first equation in (4), (3) and the well known identity

$$R^+(X,Y,Z,U) - R^-(Z,U,X,Y) = \frac{1}{2} dH(X,Y,Z,U). \tag{5}$$

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Indeed, (3) together with (5) imply

$$R^+(X,Y,Z,U) - R^-(Z,U,X,Y) = O(\alpha').$$

The first equation in (4) yields the holonomy group of ∇^+ is contained in G_2, i.e. $R^+(X,Y) \subset \mathfrak{g}_2$. Therefore R^- satisfies the instanton condition in (4) up to the first order of α'.

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Theorem (Iv-Stanchev 24)

The curvature R^- of the connection ∇^- is a G_2 instanton if and only if the torsion is closed, $dT = 0$.