# Квази-минимални повърхнини в 4-мерни псевдо-Евклидови пространства

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# Quasi-minimal (marginally trapped) surfaces

The concept of ***trapped surfaces*** – introduced by ROGER PENROSE [*Phys. Rev. Lett.*, 1965] in connection with the theory of black holes.

**Definition 1.**

A surface in the Minkowski space $\mathbb{E}^4_1$ is called ***marginally trapped*** if its mean curvature vector is lightlike at each point.

**Definition 2.**

A surface in the pseudo-Euclidean space $\mathbb{E}^4_2$ is called ***quasi-minimal*** if its mean curvature vector is lightlike at each point of the surface.

# Classification results on marginally trapped surfaces

- Marginally trapped surfaces with positive relative nullity – classified by B.-Y. Chen and J. Van der Veken [*Class. Quantum Grav.*, 2007].

The relative null space at a point $p \in M$ of a surface $M$ with second fundamental form $h$ is defined by

$$\mathcal{N}_p(M) = \{X \in T_pM \mid h(X, Y) = 0 \ \forall \ Y \in T_pM\}.$$

The dimension of $\mathcal{N}_p(M)$, denoted by $\nu_p(M)$, is called the ***relative nullity*** at $p$.
The surface $M$ is said to have ***positive relative nullity*** if $\nu_p(M) > 0$ for each $p \in M$.

# Classification results on marginally trapped surfaces

**Theorem 1.1 [Chen, Van der Veken, *Class. Quantum Grav.*, 2007]**

*Up to isometries of the Minkowski spacetime* $\mathbb{E}^4_1$*, there exist two families of marginally trapped surfaces with positive relative nullity in* $\mathbb{E}^4_1$*:*

*(1) A surface defined by* $L(u,v) = (f(u), u, v, f(u))$*, where* $f(u)$ *is an arbitrary differentiable function with* $f''(u)$ *being nowhere zero.*

*(2) A surface defined by*

$$L(u,v) = \Big(\int_0^u r(u)q'(u)du + q(u)v,\ v\cos u - \int_0^u r(u)\sin u du,$$
$$v\sin u + \int_0^u r(u)\cos u du, \int_0^u r(u)q'(u)du + q(u)v\Big),$$

*where* $q$ *and* $r$ *are defined on an open interval* $I \ni 0$ *satisfying* $q''(u) + q(u) \neq 0$ *for each* $u \in I$*.*

*Conversely, every marginally trapped surface with positive relative nullity in the Minkowski spacetime* $\mathbb{E}^4_1$ *is congruent to an open portion of a surface obtained from these two families.*

# Classification results on marginally trapped surfaces

- Marginally trapped surfaces with parallel mean curvature vector – classified by B.-Y. Chen and J. Van der Veken in [*Houston J. Math.*, 2010].

**Theorem 1.2 [Chen, Van der Veken, *Houston J. Math.*, 2010]**

*Let $M$ be a marginally trapped surface with parallel mean curvature vector in the Minkowski spacetime $\mathbb{E}^4_1$. Then, $M$ is an open part of one of the following six types of surfaces:*
*(1) a flat parallel surface given by*

$$L(u,v) = \frac{1}{2}\left((1-b)u^2 + (1+b)v^2, (1-b)u^2 + (1+b)v^2, 2u, 2v\right), \ b \in \mathbb{R};$$

*(2) a flat parallel surface given by*

$$L(u,v) = a\left(\cosh u; \sinh u; \cos v; \sin v\right), \quad a > 0;$$

# Classification results on marginally trapped surfaces

*(3) a non-parallel flat surface lying in the hyperplane* $\mathcal{H}_0 = \{(x_1; x_2; x_3; x_4) \in \mathbb{E}^4_1 : x_1 = x_4\}$ *but not in any light cone;*

*(4) a non-parallel flat surface lying in the light cone* $\mathcal{LC} = \{x = (x_1; x_2; x_3; x_4) \in \mathbb{E}^4_1 : \langle x, x \rangle = 0\}$*;*

*(5) a non-parallel surface lying in the de Sitter spacetime* $\mathbb{S}^3_1(r^2)$ *for some* $r > 0$ *such that the mean curvature vector* $H'$ *of* $M$ *in* $\mathbb{S}^3_1(r^2)$ *satisfies* $\langle H', H' \rangle = -r^2$*;*

*(6) a non-parallel surface lying in the hyperbolic space* $\mathbb{H}^3(-r^2)$ *for some* $r > 0$ *such that the mean curvature vector* $H'$ *of* $M$ *in* $\mathbb{H}^3(-r^2)$ *satisfies* $\langle H', H' \rangle = r^2$*.*

*Conversely, all surfaces of types (1)–(6) above give rise to marginally trapped surfaces with parallel mean curvature vector in* $\mathbb{E}^4_1$*.*

# Classification results on marginally trapped surfaces

- Marginally trapped surfaces in $\mathbb{E}^4_1$ which are invariant under boost transformations – in [S. HAESEN, M. ORTEGA, *Class. Quantum Grav.*, 2007].
- Marginally trapped surfaces in $\mathbb{E}^4_1$ which are invariant under spacelike rotations – in [S. HAESEN, M. ORTEGA, *Gen. Relativ. Grav.*, 2009].
- Marginally trapped surfaces which are invariant under the group of screw rotations – in [S. HAESEN, M. ORTEGA, *J. Math. Anal. Appl.*, 2009].
- Marginally trapped surfaces with pointwise 1-type Gauss map – in [V.M., *Int. J. Geom.* 2013] and [N. Turgay, *Gen. Relativ. Grav.* 2014].

## Theorem 1.3 [G. Ganchev, V.M., *J. Math. Phys.* 2012]

*Let* $\gamma_1$, $\gamma_2$, $\nu$, $\lambda$, $\mu$, $\beta_1, \beta_2$ *be smooth functions, defined in a domain* $\mathcal{D}$, $\mathcal{D} \subset \mathbb{R}^2$, *and satisfying the conditions*

$$\frac{\mu_u}{\mu(2\,\gamma_2 + \beta_1)} > 0; \qquad \frac{\mu_v}{\mu(2\,\gamma_1 + \beta_2)} > 0;$$

$$-\gamma_1\sqrt{E}\sqrt{G} = (\sqrt{E})_v; \qquad -\gamma_2\sqrt{E}\sqrt{G} = (\sqrt{G})_u;$$

$$2\lambda\,\mu = \frac{1}{\sqrt{E}}\,(\gamma_2)_u + \frac{1}{\sqrt{G}}\,(\gamma_1)_v - \left((\gamma_1)^2 + (\gamma_2)^2\right);$$

$$2\lambda\,\gamma_2 - 2\nu\,\gamma_1 - \lambda\,\beta_1 + (1+\nu)\,\beta_2 = \frac{1}{\sqrt{E}}\,\lambda_u - \frac{1}{\sqrt{G}}\,\nu_v;$$

$$2\lambda\,\gamma_1 + 2\nu\,\gamma_2 + (1-\nu)\,\beta_1 - \lambda\,\beta_2 = \frac{1}{\sqrt{E}}\,\nu_u + \frac{1}{\sqrt{G}}\,\lambda_v;$$

$$\gamma_1\,\beta_1 - \gamma_2\,\beta_2 + 2\nu\,\mu = -\frac{1}{\sqrt{E}}\,(\beta_2)_u + \frac{1}{\sqrt{G}}\,(\beta_1)_v,$$

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*where* $\sqrt{E} = \dfrac{\mu_u}{\mu(2\,\gamma_2 + \beta_1)}$, $\sqrt{G} = \dfrac{\mu_v}{\mu(2\,\gamma_1 + \beta_2)}$.

*Let* $\{x_0,\, y_0,\, (n_1)_0,\, (n_2)_0\}$ *be vectors at a point* $p_0 \in \mathbb{R}^4_1$, *such that* $x_0$, $y_0$ *are unit spacelike vectors,* $\langle x_0, y_0\rangle = 0$, $(n_1)_0, (n_2)_0$ *are lightlike vectors, and* $\langle (n_1)_0, (n_2)_0\rangle = -1$. *Then there exist a subdomain* $\mathcal{D}_0 \subset \mathcal{D}$ *and a unique marginally trapped surface* $M^2 : z = z(u,v),\ (u,v) \in \mathcal{D}_0$ *free of flat points, such that* $M^2$ *passes through* $p_0$, *the functions* $\gamma_1,\, \gamma_2,\, \nu,\, \lambda,\, \mu,\, \beta_1, \beta_2$ *are the geometric functions of* $M^2$ *and* $\{x_0,\, y_0,\, (n_1)_0,\, (n_2)_0\}$ *is the geometric frame of* $M^2$ *at the point* $p_0$.

# Classification results on quasi-minimal surfaces in $\mathbb{E}^4_2$

- In [*J. Math. Anal. Appl.*, 2008] B.-Y. Chen classified quasi-minimal Lorentz flat surfaces in $\mathbb{E}^4_2$.
- Quasi-minimal surfaces with constant (non-zero) Gauss curvature in $\mathbb{E}^4_2$ were classified by B.-Y. Chen and D. Yang in [*Hokkaido Math. J.*, 2009] and [*J. Math. Anal. Appl.*, 2010].
- The classification of quasi-minimal surfaces with parallel mean curvature vector is obtained in [Chen, Garay, *Result. Math.*, 2009].
- In [*Cent. Eur. J. Math.*, 2014] G. Ganchev and V. M. obtained the classification of quasi-minimal rotational surfaces in $\mathbb{E}^4_2$.

# Quasi-minimal Rotational Surfaces in $\mathbb{E}^4_2$

## Theorem 1.4 [G. Ganchev, V.M., *Cent. Eur. J. Math.*, 2014]

*Given a smooth positive function $r(u) : I \subset \mathbb{R} \to \mathbb{R}$, define the functions*

$$\varphi(u) = \eta \int \frac{rr'' + (r')^2 + 1}{r(1 + (r')^2)}\, du, \quad \eta = \pm 1,$$

*and*

$$x_1(u) = \int \sqrt{1 + (r')^2}\, \cos\varphi(u)\, du,$$
$$x_2(u) = \int \sqrt{1 + (r')^2}\, \sin\varphi(u)\, du.$$

*Then the spacelike curve $c : \widetilde{z}(u) = (x_1(u), x_2(u), r(u), 0)$ is a generating curve of a quasi-minimal rotational surface of elliptic type.*

*Conversely, any quasi-minimal rotational surface of elliptic type is locally constructed as above.*

# Quasi-minimal Rotational Surfaces in $\mathbb{E}^4_2$

**Theorem 1.5 [G. Ganchev, V.M., *Cent. Eur. J. Math.*, 2014]**

*Given a smooth function $f(u) : I \subset \mathbb{R} \to \mathbb{R}$, define the functions*

$$\varphi(u) = f'(u)\left(C + \eta\left(-\frac{1}{f'(u)} + \int \frac{du}{f(u)}\right)\right), \quad \eta = \pm 1, \; C = const,$$

*and*

$$x_1(u) = \int \varphi(u)du; \qquad g(u) = \int \frac{\varphi^2(u) - 1}{2f'(u)}\, du.$$

*Then the curve $c : \widetilde{z}(u) = x_1(u)\, e_1 + f(u)\, \xi_1 + g(u)\, \xi_2$ is a spacelike curve generating a quasi-minimal rotational surface of parabolic type.*

*Conversely, any quasi-minimal rotational surface of parabolic type is locally constructed as described above.*

# Quasi-minimal surfaces with pointwise 1-type Gauss map

## Definition

A submanifold $M$ of the Euclidean space $\mathbb{E}^m$ (or pseudo-Euclidean space $\mathbb{E}^m_s$) is said to have ***pointwise 1-type Gauss map*** if its Gauss map $G$ satisfies

$$\Delta G = \phi(G + C)$$

for some non-zero function $\phi$ on $M$ and a constant vector $C$.

A pointwise 1-type Gauss map is called ***proper*** if the function $\phi$ is non-constant.

A submanifold with pointwise 1-type Gauss map is said to be of ***first kind*** if the vector $C$ is zero. Otherwise, the pointwise 1-type Gauss map is said to be of ***second kind***.

**Theorem 1.6. [V.M., N.C. Turgay, *J. Geom. Phys.*]**

*Let $M^2_1$ be a* ***flat quasi-minimal surface*** *in the pseudo-Euclidean space $\mathbb{E}^4_2$. Then, $M^2_1$ has pointwise 1-type Gauss map if and only if it is congruent to the surface given by*

$$z(u,v) = \left(\theta(u,v), \frac{u-v}{\sqrt{2}}, \frac{u+v}{\sqrt{2}}, \theta(u,v)\right)$$

*for a smooth function $\theta$.*

# Non-flat quasi-minimal surfaces with flat normal connection

**Theorem 1.7 [V.M., N.C. Turgay, *J. Geom. Phys.*]**

*Let $M_1^2$ be a **quasi-minimal** surface in the pseudo-Euclidean space $\mathbb{E}_2^4$ with **flat normal connection** and **non-vanishing Gauss curvature**. Then, $M_1^2$ has **pointwise 1-type Gauss map** if and only if it belongs to one of the following two families:*

(i) *a non-flat CMC-surface lying in $\mathbb{S}_2^3(r^2)$ for some $r > 0$ such that the mean curvature vector $H'$ of $M$ in $\mathbb{S}_2^3(r^2)$ satisfies $\langle H', H' \rangle = -r^2$;*

(ii) *a non-flat CMC-surface lying in $\mathbb{H}_1^3(-r^2)$ for some $r > 0$ such that the mean curvature vector $H'$ of $M$ in $\mathbb{H}_2^3(-r^2)$ satisfies $\langle H', H' \rangle = r^2$.*

# Quasi-minimal surfaces with non-flat normal connection

## Theorem 1.8. [V.M., N.C. Turgay, *J. Geom. Phys.*]

*Let $M_1^2$ be a* ***quasi-minimal*** *surface in $\mathbb{E}_2^4$ with* ***non-flat normal connection****. Then, $M_1^2$ has pointwise 1-type Gauss map if and only if it is congruent to the surface given by*

$$z(s,t) = -s\lambda_3(t)n_1'(t) - \frac{3\sqrt{6}\lambda_3(t)\sqrt{-s\lambda_1(t)}}{\lambda_1^2(t)}n_1(t) + \xi(t)$$

*for some smooth functions $\lambda_1 = \lambda_1(t),\ \lambda_3 = \lambda_3(t)$ and some $\mathbb{E}_2^4$-valued smooth functions $n_1(t)$, $\xi(t)$ satisfying the equations*

$$\langle n_1, n_1\rangle = \langle n_1', n_1'\rangle = \langle n_1, \xi'\rangle = \langle \xi', \xi'\rangle = 0, \qquad \langle n_1', \xi'\rangle = \frac{1}{\lambda_3},$$

$$n_1'' - \left(\frac{\lambda_3'}{\lambda_3} - \frac{3\lambda_1'}{\lambda_1}\right)n_1' + \frac{1}{\lambda_3}n_1 = 0,$$

*and*

$$\xi''' + \left(\frac{3\lambda_3'}{\lambda_3} - \frac{3\lambda_1'}{\lambda_1}\right)\xi'' + \frac{-3\lambda_1\left(\lambda_1'\lambda_3' + \lambda_3\lambda_1''\right) + 3\lambda_3{\lambda_1'}^2 + \lambda_1^2\left(2\lambda_3'' + 1\right)}{\lambda_1^2\lambda_3}\,\xi' = \zeta,$$

*where* $\zeta = \zeta(t)$ *is the* $\mathbb{E}^4_2$*-valued function given by*

$$\zeta = \frac{8{\lambda_1'}^2\lambda_3^2 - 2\lambda_1\lambda_1''\lambda_3^2 + \lambda_1^2\lambda_3\lambda_3'' - 7\lambda_1\lambda_1'\lambda_3\lambda_3' + \lambda_1^2{\lambda_3'}^2}{81^{-1}\lambda_1^5\lambda_3}\,n_1 + \frac{\lambda_1\lambda_3' - 2\lambda_3\lambda_1'}{162^{-1}\lambda_1^4}\,n_1'.$$

# Example 1.

Let $\mathcal{M}$ be the surface given by the following parametrization

$$z(s,t) = (-4s^{1/2}\cos t + s\sin t + \tfrac{1}{2}\cos t)e_1 - (4s^{1/2}\sin t + s\cos t - \tfrac{1}{2}\sin t)$$
$$-(4s^{1/2}\sin t + s\cos t + \tfrac{1}{2}\sin t)e_3 + (-4s^{1/2}\cos t + s\sin t - \tfrac{1}{2}\cos t)e_4.$$

$\mathcal{M}$ is a **quasi-minimal Lorentz surface** in $\mathbb{E}^4_2$.

$$K = s^{-3/2}, \quad \varkappa = s^{-3/2}.$$

$\mathcal{M}$ is a surface with **non-flat normal connection** and **non-parallel mean curvature vector field**.

# General Fundamental Theorem for quasi-minimal surfaces

## Theorem 1.9

Let $\lambda_1(u,v)$, $\mu_1(u,v)$, $\lambda_2(u,v)$, $\mu_2(u,v)$, $f(u,v)$ be smooth functions, defined in a domain $\mathcal{D}$, $\mathcal{D} \subset \mathbb{R}^2$, and satisfying the conditions

$$\begin{aligned}
& f > 0; \qquad \lambda_1\mu_1\lambda_2\mu_2 \neq 0, \quad (u,v) \in \mathcal{D}; \\
& \left(\ln|\lambda_1\mu_1 f^4|\right)_v = \tfrac{1}{\lambda_1}\left(\ln|\mu_2 f^2|\right)_u; \\
& \left(\ln|\lambda_2\mu_2 f^4|\right)_u = \tfrac{1}{\lambda_2}\left(\ln|\mu_1 f^2|\right)_v; \\
& 2ff_{uv} - 2f_u f_v = f^4(\lambda_1\mu_2 + \mu_1\lambda_2); \\
& \left(\ln\left|\tfrac{\mu_1}{\mu_2}\right|\right)_{uv} = f^2(\lambda_1\mu_2 - \mu_1\lambda_2).
\end{aligned} \tag{1}$$

Let $\{x_0,\, y_0,\, (n_1)_0,\, (n_2)_0\}$ be a pseudo-orthonormal frame at a point $p_0 \in \mathbb{E}^4_2$. Then, there exist a subdomain $\mathcal{D}_0 \subset \mathcal{D}$ and a unique quasi-minimal Lorentz surface $M^2_1 : z = z(u,v),\ (u,v) \in \mathcal{D}_0$ passing through the point $p_0$.

# Example 2.

Let $\mathcal{M}$ be a surface in the Minkowski 4-space given by:

$$\mathcal{M} : z(u,v) = \left(1 + \frac{u^2+v^2}{2}\right) e_1 + \frac{u^2+v^2}{2} e_2 + u\, e_3 + v\, e_4.$$

By direct computation it can be shown that $\mathcal{M}$ is a **quasi-minimal (marginally trapped) surface** in $\mathbb{E}^4_1$ with **parallel normal bundle**.

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