On a dispersive system

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- nonlinear dispersive system (DSW)

$$
\left|\begin{array}{l} u_t + v v_x = 0 \\ \\ v_t + v_{xxx} + u v_x + u_x v = 0, \end{array}\right. \tag{1}
$$

where are u and v are real valued functions.

- We look for solutions in the form $u(t,x) = \phi(x - ct)$ and $v(t,x) = \psi(x - ct)$. Plugging in the system and integrating once, we get

$$\begin{cases} \phi = \frac{1}{2c}\psi^2 + \frac{A}{c} \\ -c\psi + \psi'' + \phi\psi = B, \end{cases} \tag{2}$$

Replacing $\phi = \frac{1}{2c}\psi^2$ in the second equation for ψ, we get the equation

$$\psi'' = -\frac{1}{2c}\psi^3 + c\psi + B. \tag{3}$$

Here, we consider the symmetric case $B = 0$. After integrating, we get

$$\psi'^2 = -\frac{1}{4c}\psi^4 + c\psi^2 + a, \tag{4}$$

where a is a constant of integration.

- cn solutions
  For $c > 0$ and $a > 0$, denote by $\psi_0, -\psi_0, i\psi_1, -i\psi_1$ the roots of polynomial $-r^4 + 4c^2 r^2 + 4ca$. Then, the equation (4) can be written in the form

$$\psi'^2 = \frac{1}{4c}(\psi_0^2 - \psi^2)(\psi_1^2 + \psi^2) = \frac{1}{4c}(\psi_0^2 - \psi^2)(\psi_0^2 - 4c^2 + \psi^2).$$

The solution is given by

$$\psi(x) = \psi_0 cn(\alpha x, \kappa), \tag{5}$$

where

$$\kappa^2 = \frac{\psi_0^2}{2\psi_0^2 - 4c^2} \quad \alpha^2 = \frac{\psi_0^2 - 2c^2}{2c} = \frac{c}{2\kappa^2 - 1}. \tag{6}$$

Since the fundamental period of $cn(x)$ is $4K(\kappa)$, then the fundamental period of $\psi(x)$ is $2T = \frac{4K(\kappa)}{\alpha}$.

- Consider the perturbations in the form $u(t,x) = \phi(x-ct) + p(t,x-ct)$ and $v(t,x) = \psi(x-ct) + q(t,x-ct)$. Plugging in the system (1) and ignoring all quadratic and higher order terms, we get the following linear equation for (p,q)

$$\vec{U}_t = \mathcal{J}\mathcal{H}\vec{U}, \tag{7}$$

where

$$\vec{U} = \begin{pmatrix} p \\ q \end{pmatrix} \quad \mathcal{J} = \begin{pmatrix} \partial_x & 0 \\ 0 & \partial_x \end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix} c & -\psi \\ -\psi & L_1 \end{pmatrix}. \tag{8}$$

and

$$L_1 = -\partial_x^2 + c - \frac{1}{2c}\psi^2.$$

- Clearly $\mathcal{J}^* = -\mathcal{J}$, whereas $\mathcal{H}^* = \mathcal{H}$, with the natural domains on the periodic functions , namely

$$\begin{aligned} D(\mathcal{J}) &= (H^1[-T, T]) \oplus (H^1[-T, T]) \\ D(\mathcal{H}) &= (L^2[-T, T]) \oplus (H^2[-T, T]) \end{aligned}$$

The standard mapping $\begin{pmatrix} U \\ V \end{pmatrix} \to e^{\lambda t} \vec{z}$ transforms the linear differential equation (7) into the eigenvalue problem

$$\mathcal{J}\mathcal{H}\vec{z} = \lambda \vec{z}. \tag{9}$$

Definition

We say that the waves (ϕ, ψ) are spectrally unstable, if the eigenvalue problem (9) has a non-trivial solution $(\vec{z}, \lambda)$, so that $\vec{z} \neq 0, \vec{z} \in L^2[-T, T] \times H^2[-T, T]$ and $\lambda : \Re\lambda > 0$.
In the opposite case, that is (9) has no non-trivial solutions, with $\Re\lambda > 0$, we say that the waves are spectrally stable.

- Stability of periodic traveling waves of cnoidal type
  Let ψ be a cnoidal wave defined by (5). Then, the kernel of $\mathcal{H}$ is one dimensional ($\dim\ker\mathcal{H} = 1$) and spanned by $\Psi_1 = \begin{pmatrix} \frac{1}{c}\psi\psi' \\ \psi' \end{pmatrix}$.
  We have that $\begin{pmatrix} f \\ g \end{pmatrix} \in \ker\mathcal{H}$, if

$$\left| \begin{array}{l} cf - \psi g = 0 \\ \\ -\psi f + L_1 g = 0. \end{array} \right.$$

  From the first equation, we have $f = \frac{1}{c}\psi g$ and plagging in the second equation, we get

$$\left(L_1 - \frac{1}{c}\psi^2\right) g = \mathcal{L}g = 0. \tag{10}$$

- the kernel of $\mathcal{L} = -\partial_x^2 + c - \frac{3}{2c}\psi^2$ is one dimensional and spanned by ψ'. Hence the all solutions of $\mathcal{H}\begin{pmatrix} f \\ g \end{pmatrix} = 0$ are multiples of $g = \psi'$ and $f = \frac{1}{c}\psi\psi'$.

- number of negative eigenvalues of the operator $\mathcal{H}$. If ψ is cnoidal wave given by (5), then $n(\mathcal{H}) = 2$. Consider the quadratic form associated to $\mathcal{H}$, namely

$$\begin{aligned} \langle \mathcal{H} \begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix} \rangle &= \langle f, f \rangle - \langle \psi g, f \rangle + \langle L_1 g, g \rangle - \langle \psi f, g \rangle \\ &= \langle \mathcal{L} g, g \rangle + \int_{-T}^{T} \left(\sqrt{c} f - \frac{1}{\sqrt{c}} \psi g \right)^2 dx. \end{aligned} \tag{11}$$

- $n(\mathcal{L}) = 2$

$$\inf_{\begin{pmatrix} f \\ g \end{pmatrix} \perp \begin{pmatrix} 0 \\ \chi_0 \end{pmatrix}, \begin{pmatrix} 0 \\ \chi_1 \end{pmatrix}} \langle \mathcal{H} \begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix} \rangle \geq \inf_{g \perp \chi_0, \chi_1} \langle \mathcal{L} g, g \rangle \geq 0.$$

By the Rayleigh-Ritz formulas, it follows that the third smallest eigenvalue of $\mathcal{H}$ is non-negative and hence $n(\mathcal{H}) \leq 2$.
Take $g_0 = \chi_0$ and $f_0 = \frac{1}{c}\psi\chi_0$, We have

$$\langle \mathcal{H} \begin{pmatrix} f_0 \\ g_0 \end{pmatrix}, \begin{pmatrix} f_0 \\ g_0 \end{pmatrix} \rangle = \langle \mathcal{H} \begin{pmatrix} \frac{1}{c}\psi\chi_0 \\ \chi_0 \end{pmatrix}, \begin{pmatrix} \frac{1}{c}\psi\chi_0 \\ \chi_0 \end{pmatrix} \rangle = \langle \mathcal{L}\chi_0, \chi_0$$

$$= \lambda_0 \langle \chi_0, \chi_0 \rangle < 0.$$

Hence, $n(\mathcal{H}) \geq 1$.

In order to show that $n(\mathcal{H}) = 2$ we need to construct second vector $\begin{pmatrix} f_1 \\ g_1 \end{pmatrix} \perp \begin{pmatrix} f_0 \\ g_0 \end{pmatrix}$, such that $\langle \mathcal{H} \begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, \begin{pmatrix} f_1 \\ g_1 \end{pmatrix} \rangle < 0$.
Take $g_1 = \chi_1$ and $f_1 = \frac{1}{c}\psi\chi_1$.
It follows that $n(\mathcal{H}) = 2$.
If $2cT + \langle \mathcal{L}^{-1}\psi, \psi \rangle \neq 0$ and $\langle \mathcal{L}^{-1}[1], 1 \rangle \neq 0$, and $\langle \mathcal{L}^{-1}\psi, 1 \rangle = 0$, then $g \ker(\mathcal{J}\mathcal{H}) \ominus \ker \mathcal{H}$ is three dimensional and spanned by the vectors

$$\eta_1 = \begin{pmatrix} 1 \\ \\ \frac{1}{2c}\psi \end{pmatrix}, \quad \eta_2 = \begin{pmatrix} \frac{1}{c} + \frac{1}{c^2}\psi\mathcal{L}^{-1}\psi \\ \\ \frac{1}{c}\mathcal{L}^{-1}\psi \end{pmatrix}, \quad \eta_3 = \begin{pmatrix} \frac{1}{c}\psi\mathcal{L}^{-1}[1] \\ \\ \mathcal{L}^{-1}[1] \end{pmatrix}.$$

- Generalized kernel of $\mathcal{J}\mathcal{H}$

$$gKer(\mathcal{J}\mathcal{H}) = span[(Ker(\mathcal{J}\mathcal{H}))^l, l = 1, 2, \ldots].$$

Select a basis in $gKer(\mathcal{J}\mathcal{H}) \ominus Ker(\mathcal{H}) = span[\eta_j, j = 1, \ldots]$. Then $\mathcal{D}$ is defined via

$$\mathcal{D} := \{\mathcal{D}_{ij}\}_{i,j=1}^{N} : \mathcal{D}_{ij} = \langle \mathcal{H}\eta_i, \eta_j \rangle.$$

$$k_{Ham} := k_r + 2k_c + 2k_i^- = n(\mathcal{H}) - n_0(\mathcal{D}), \qquad (12)$$

representation for matrix $\mathcal{D}$,

$$\mathcal{D} = \begin{pmatrix} \mathcal{D}_{11} & \mathcal{D}_{12} & 0 \\ \mathcal{D}_{12} & \mathcal{D}_{22} & 0 \\ 0 & 0 & \mathcal{D}_{33} \end{pmatrix}$$

Hence the number of non-positive eigenvalues the matrix $\mathcal{D}$ is $n_0(\mathcal{D}) = n_0(\mathcal{D}_0) + n_0(\mathcal{D}_{33})$, where $\mathcal{D}_0 = \begin{pmatrix} \mathcal{D}_{11} & \mathcal{D}_{11} \\ \mathcal{D}_{12} & \mathcal{D}_{22} \end{pmatrix}$.

Since we require that $\langle \mathcal{L}^{-1}[1], 1 \rangle \neq 0$, then $n_0(\mathcal{D}_{33}) = n(\mathcal{D}_{33})$.

The matrix $\mathcal{D}_0$ has a one negative and one positive eigenvalues.

$$n(\mathcal{D}_{33}) = n(\langle \mathcal{L}^{-1}[1], 1\rangle) = \begin{cases} 1, & \frac{1}{\sqrt{2}} < \kappa < \kappa^* \\ 0, & \kappa^* < \kappa < 1. \end{cases}$$

and whence

$$n_0(\mathcal{D}) = \begin{cases} 2, & \frac{1}{\sqrt{2}} < \kappa < \kappa^* \\ 1, & \kappa^* < \kappa < 1 \end{cases} \tag{13}$$

With this we get that,

$$k_{Ham} := k_r + 2k_c + 2k_i^- = n(\mathcal{H}) - n_0(\mathcal{D}) = \begin{cases} 0, & \frac{1}{\sqrt{2}} < \kappa < \kappa^* \\ 1, & \kappa^* < \kappa < 1 \end{cases} .$$

Theorem

There is κ^*, such that periodic traveling waves (ϕ, ψ), where ψ is given by (5) are spectrally stable for $\frac{1}{\sqrt{2}} < \kappa < \kappa^*$ and unstable for $\kappa^* < \kappa < 1$.

Thank you for Attention