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ВЪРХУ ЕДИН КЛАС МАЛКО ИЗВЕСТНИ, НО ВАЖНИ СПЕЦИАЛНИ I-ФУНКЦИИ

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ON A CLASS OF NOT WELL KNOWN, BUT IMPORTANT SPECIAL I-FUNCTIONS

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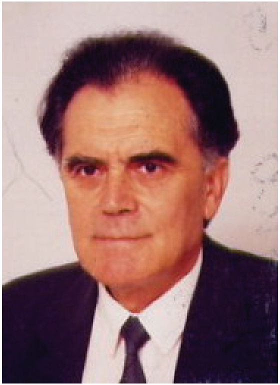
To the memory of our teachers:
Profs. Peter Rusev and Ivan Dimovski

Definition. The Fox H-function (Ch. Fox (1961) is a generalized hypergeometric function, defined by means of the Mellin-Barnes type contour integral in the complex plane

$$H^{m,n}_{p,q}\left[z \,\middle|\, \begin{array}{l}(a_i,A_i)_1^p \\ (b_j,B_j)_1^q\end{array}\right] = \frac{1}{2\pi i}\int_{\mathcal{L}} \mathcal{H}^{m,n}_{p,q}(s)\, z^{-s} ds, \quad z \neq 0, \qquad (1)$$

$$\text{where} \quad \mathcal{H}^{m,n}_{p,q}(s) = \frac{\prod\limits_{j=1}^{m}\Gamma(b_j+B_js)\prod\limits_{i=1}^{n}\Gamma(1-a_i-A_is)}{\prod\limits_{j=m+1}^{q}\Gamma(1-b_j-B_js)\prod\limits_{i=n+1}^{p}\Gamma(a_i+A_is)},$$

with a contour $\mathcal{L}$ (3 types); the orders (m,n,p,q) non negative integers so that $0 \le m \le q$, $0 \le n \le p$, the parameters $A_i > 0, B_j > 0$ are positive, and a_i, b_j, $i = 1,\ldots,p; j = 1,\ldots,q$ are arbitrary complex such that $A_i(b_j+l) \neq B_j(a_i-l'-1)$, $l, l' = 0, 1, 2, \ldots;\ i = 1,\ldots,n;\ j = 1,\ldots,m$. Details on the types of contours $\mathcal{L}$ and properties of the H-function – in many contemporary handbooks on SF, where its behaviour is described via the parameters: R, Δ, ∇, μ, ... For $\forall A_i = B_j = 1$, the H-function reduces to a Meijer G-function, then to the classical SF ("SF of Mathematical Physics").

Most of the SF of FC, being H-functions, are practically cases of the Wright (Fox-Wright) generalized hypergeometric functions

$$ {}_p\Psi_q \left[\begin{array}{c} (a_1,A_1),...,(a_p,A_p) \\ (b_1,B_1),...,(b_q,B_q) \end{array} \middle| z \right] = \sum_{k=0}^{\infty} \frac{\Gamma(a_1+kA_1)\dots\Gamma(a_p+kA_p)}{\Gamma(b_1+kB_1)\dots\Gamma(b_q+kB_q)} \frac{z^k}{k!} $$

$$ = H^{1,p}_{p,q+1}\left[-z \middle| \begin{array}{c} (1-a_1,A_1),\dots,(1-a_p,A_p) \\ (0,1),(1-b_1,B_1),\dots,(1-b_q,B_q) \end{array} \right]. \qquad (2) $$

Denote: $R=\prod_{i=1}^{p} A_i^{-A_i} \prod_{j=1}^{q} B_j^{B_j}$, $\Delta=\sum_{k=1}^{j} B_j - \sum_{i=1}^{p} A_i$. If $\Delta>-1$, the ${}_p\Psi_q$-function is an entire function of $z\in\mathbb{C}$, but if $\Delta=-1$, the series is absolutely convergent in the disk $\{|z|<R\}$, while ....

If all $A_1=\cdots=A_p=1, B_1=\cdots=B_q=1$, the Wright g.h.f. reduces to the *generalized hypergeometric ${}_pF_q$-function*, which itself is a case of the Meijer G-function,

$$ {}_p\Psi_q \left[\begin{array}{c} (a_1,1),\dots,(a_p,1) \\ (b_1,1),\dots,(b_q,1) \end{array} \middle| z \right] = c\, {}_pF_q(a_1,\dots,a_p; b_1,\dots,b_q; z) $$

$$ = \sum_{k=0}^{\infty} \frac{(a_1)_k\dots(a_p)_k}{(b_1)_k\dots(b_q)_k} \frac{z^k}{k!} = G^{1,p}_{p,q+1}\left[-z \middle| \begin{array}{c} 1-a_1,...,1-a_p \\ 0,1-b_1,...,1-b_q \end{array} \right]; \; c=\dots $$

The Mittag-Leffler (ML) functions, when $\alpha > 0$ is not integer or rational index, are the simplest examples of ${}_p\Psi_q$-functions that are *not* ${}_pF_q$- and G-functions,

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}; \;\; \alpha = 1 : E_\alpha(z), \tag{3}$$

$$E^{\tau}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\tau)_k}{\Gamma(\alpha k + \beta)} \frac{z^k}{k!} \;\; \text{(Prabhkar f.)}. \tag{4}$$

And the multi-index ML functions (Kiryakova, 1996 $\rightarrow$ / also Luchko)

$$E_{(\alpha_i),(\beta_i)} = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha_1 k+\beta_1)\dots\Gamma(\alpha_m k+\beta_m)}, \tag{5}$$

and (Paneva-Konovska, 2011 $\rightarrow$)

$$E^{(\tau_i),m}_{(\alpha_i),(\beta_i)}(z) = \sum_{k=0}^{\infty} \frac{(\tau_1)_k \dots (\tau_m)_k}{\Gamma(\alpha_1 k+\beta_1)\dots\Gamma(\alpha_m k+\beta_m)} \frac{z^k}{(k!)^m}, \tag{6}$$

include almost all the SF of FC (appearing as solutions of FO models). A very long list of such SF has been provided as evidence in our survey papers.

However, another SF related to FC, the Le Roy function (1900) attracted the attention of our FC colleagues *ONLY* recently,

$$\exp(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!} \longrightarrow \text{(Le Roy) } F_\gamma(z) = \sum_{k=0}^{\infty} \frac{z^k}{(k!)^\gamma}$$

and together with newly introduced their ML-type analogues:

$$\longrightarrow \text{(MLR: Gerhold, Garra-Polito) } F_{\alpha,\beta}^{(\gamma)}(z) = \sum_{k=0}^{\infty} \frac{z^k}{[\Gamma(\alpha k + \beta)]^\gamma}$$

$$(\gamma > 0, \gamma_i > 0) \longrightarrow \text{(3}m\text{-MLR: Rogosin) } F_{(\alpha,\beta)_m}^{(\gamma)_m}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\prod_{i=1}^{m} [\Gamma(\alpha_i k + \beta_i)]^{\gamma_i}}$$

happen NOT to be H-functions and ${}_p\Psi_q$-functions !

The above SF have been studied, since 2012 to 2021, by: Gerhold, Garra, Polito, Garrappa, Mainardi, Rogosin, Orsingher, Gorska, Horzela, Simon, etc.

Then, from our 3m-multi-ML and the above 3m-MLR functions, since 2022, we went further to a new class of SF of Le Roy type, as follows:

Definition. Multi-index Mittag-Leffler-Prabhakar functions of Le Roy type (multi-MLPR), suppose all the $4m$ parameters are > 0:

$$\mathbb{F}_m(z) := \mathbb{F}^{\gamma_i;m}_{\alpha_i,\beta_i;\tau_i}(z) \tag{7}$$

$$= \sum_{k=0}^{\infty} \frac{(\tau_1)_k \ldots (\tau_m)_k}{(k!)^m} \cdot \frac{z^k}{[\Gamma(\alpha_1 k + \beta_1)]^{\gamma_1} \ldots [\Gamma(\alpha_m k + \beta_m)]^{\gamma_m}}$$

$$= \sum_{k=0}^{\infty} c_k\, z^k, \ \text{ with } \ c_k = \prod_{i=1}^{m} \left\{ \frac{\Gamma(k+\tau_i)}{\Gamma(k+1)} \cdot \frac{1}{\Gamma(\tau_i)} \cdot \frac{1}{[\Gamma(\alpha_i k+\beta_i)]^{\gamma_i}} \right\}.$$

Several basic analytical properties of (7) are proposed in our recent papers: 2022-2023 by VK-JPK + Rogosin-Dubatovskaya, 2022-2024 by VK-JPK.

Theorem. (2023) *Suppose*

$\forall i = 1, ..., m : \alpha_i > 0, \beta_i > 0, \gamma_i > 0, \tau_i > 0,$ *and* $\Rightarrow \sum_{i=1}^{m} \alpha_i\gamma_i > 0.$ *The multi-index MLPR-function* (7) *is an entire function of the complex variable* z *of order* ρ *and type* σ*:*

$$\rho = \frac{1}{\alpha_1\gamma_1 + \cdots + \alpha_m\gamma_m}, \ \ \textit{and} \ \ \sigma = \frac{1}{\rho} \left(\prod_{i=1}^{m} (\alpha_i)^{-\alpha_i\gamma_i} \right)^{\rho}. \tag{8}$$

In Sonix, one can find a list of our joint (VK-JPK) and self-authored (VK, JPK) indexed (Scopus - WoS) publications and talks (abroad and in BG) on the results

for this new class of SFs and their relations to the I-functions (commented on next slides), in the last 2 years:

2023: 6 papers / 5 talks
2024: 5 papers / 8 talks

and their citations.

Definition. The so-called I-function was defined by Rathie (1997), by means of a kind of Mellin-Barnes type integral

$$I_{p,q}^{m,n}\left[z\left|\begin{array}{l}(a_i,A_i,\alpha_i)_1^p\\(b_j,B_j,\beta_j)_1^q\end{array}\right.\right]=\frac{1}{2\pi i}\int_{\mathcal{L}}\mathcal{I}_{p,q}^{m,n}(s)\,z^{-s}ds,\quad z\neq 0,$$

$$\text{with}\quad \mathcal{I}_{p,q}^{m,n}(s)=\frac{\prod_{j=1}^{m}\Gamma^{\beta_j}(b_j+B_js)\prod_{i=1}^{n}\Gamma^{\alpha_i}(1-a_i-A_is)}{\prod_{j=m+1}^{q}\Gamma^{\beta_j}(1-b_j-B_js)\prod_{i=n+1}^{p}\Gamma^{\alpha_i}(a_i+A_is)}. \tag{9}$$

Note that if $\forall\alpha_i=\forall\beta_j=1$, $i=1,\ldots,p$, $j=1,\ldots,q$, this is the Fox H-function. But in general, these are NOT positive integers. Then, we have multi-valued functions Γ whose singularities are now branch points. Some more simple case of this SF, is the $\overline{H}$-function of Inayat-Hussain (1987), where in particular some of the α_i,β_j are equal to 1, namely: $\alpha_i=1, i=n+1,\ldots,p$ and $\beta_j=1, j=1,\ldots,m$:

$$\overline{H}_{p,q}^{m,n}\left[z\left|\begin{array}{l}(a_i,A_i,\alpha_i)_1^p\\(b_j,B_j,\beta_j)_1^q\end{array}\right.\right]=\frac{1}{2\pi i}\int_{\mathcal{L}}\frac{\prod_{j=1}^{m}\Gamma^{\mathbf{1}}(b_j+B_js)\prod_{i=1}^{n}\Gamma^{\alpha_i}(1-a_i-A_is)}{\prod_{j=m+1}^{q}\Gamma^{\beta_j}(1-b_j-B_js)\prod_{i=n+1}^{p}\Gamma^{\mathbf{1}}(a_i+A_is)}z^{-s}ds.$$

In our very recent papers (2024), we have introduced a generalization of the Fox-Wright function

$$ {}_p\Psi_q \left[\begin{array}{c} (a_1, A_1), ..., (a_p, A_p) \\ (b_1, B_1), ..., (b_q, B_q) \end{array} \middle| z \right] = \sum_{k=0}^{\infty} \frac{\Gamma(A_1 k + a_1)...\Gamma(A_p k + a_p)}{\Gamma(B_1 k + b_1)...\Gamma(B_q k + b_q)} \frac{z^k}{k!}. $$

Definition. We define the generalized Fox–Wright function by the power series

$$ {}_p\widetilde{\Psi}_q \left[\begin{array}{c} (a_j, A_j; \alpha_j)_{j=1}^{p} \\ (b_i, B_i; \beta_i)_{i=1}^{q} \end{array} \middle| z \right] = \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma^{\alpha_j}(A_j k + a_j)}{\prod_{i=1}^{q} \Gamma^{\beta_i}(B_i k + b_i)} \cdot \frac{z^k}{k!}, \quad (10) $$

with arbitrary real or complex parameters a_j, b_i, positive A_j, B_i and additional "fractional" power parameters $\alpha_j > 0, \beta_i > 0$, $j = 1, ..., p, i = 1, ..., q$.

The behavior and properties of this new SF are characterized by the parameters:

$$ \widetilde{\mu} = 1 + \sum_{i=1}^{q} \beta_i B_i - \sum_{j=1}^{p} \alpha_j A_j, \quad \widetilde{R} = \prod_{i=1}^{q} B_i^{\beta_i B_i} / \prod_{j=1}^{p} A_j^{\alpha_j A_j}. \quad (11) $$

Here, $\widetilde{\mu}$ indicates when the series in (10) represents an entire function, or is an analytic in $|z| < \widetilde{R}$, or converges only at $z = 0$.

Namely,

- If $\widetilde{\mu} > 0$, then the series (10) defines an entire function (that is, absolutely convergent for all $z \in \mathbb{C}$).
- If $\widetilde{\mu} = 0$, then the series (10) defines an analytical function in the open disk $|z| < \widetilde{R}$.
- If $\widetilde{\mu} < 0$, then the series (10) converges only at the point 0.

We are mostly interested in the case $\widetilde{\mu} > 0$ when ${}_p\widetilde{\Psi}_q$ is an entire function.

Theorem. (2024) *Let ${}_p\widetilde{\Psi}_q$ be the generalized Fox–Wright function* (10) *with all positive parameters a_j, A_j, α_j, b_i, B_i, β_i, and let $\widetilde{\mu} > 0$. Then, the order $\widetilde{\rho}$ and type $\widetilde{\sigma}$ of the entire function* (10) *are*

$$\frac{1}{\widetilde{\rho}} = 1 + \sum_{i=1}^{q} \beta_i B_i - \sum_{j=1}^{p} \alpha_j A_j = \widetilde{\mu}, \tag{12}$$

respectively

$$\widetilde{\sigma} = \widetilde{\mu} \left(\prod_{j=1}^{p} A_j^{\alpha_j A_j} \right)^{1/\widetilde{\mu}} \Big/ \left(\prod_{i=1}^{q} B_i^{\beta_i B_i} \right)^{1/\widetilde{\mu}}. \tag{13}$$

The generalized Fox–Wright function (10) is representable in terms of the following I- and $\overline{H}$-functions:

$$ {}_p\widetilde{\Psi}_q\left[\begin{array}{c}(a_j,A_j,\alpha_j)_{j=1}^p\\ (b_i,B_i,\beta_i)_{i=1}^q\end{array}\middle| z\right] = \overline{H}^{1,p}_{p,q+1}\left[-z\middle|\begin{array}{c}(1-a_j,A_j,\alpha_j)_1^p\\ (0,1),(1-b_i,B_i,\beta_i)_1^q\end{array}\right]$$

$$= I^{1,p}_{p,q+1}\left[-z\middle|\begin{array}{c}(1-a_j,A_j,\alpha_j)_1^p\\ (0,1,1),(1-b_i,B_i,\beta_i)_1^q\end{array}\right]. \qquad (14)$$

Then, one observes the natural parallel with the representation of the Fox–Wright function (2) by a H-function:

$$ {}_p\Psi_q\left[\begin{array}{c}(a_1,A_1),\ldots,(a_p,A_p)\\ (b_1,B_1),\ldots,(b_q,B_q)\end{array}\middle| z\right] = {}_p\widetilde{\Psi}_q\left[\begin{array}{c}(a_1,A_1,1),\ldots,(a_p,A_p,1)\\ (b_1,B_1,1),\ldots,(b_q,B_q,1)\end{array}\middle| z\right]$$

$$= H^{1,p}_{p,q+1}\left[-z\middle|\begin{array}{c}(1-a_j,A_j)_1^p\\ (0,1),(1-b_i,B_i)_1^q\end{array}\right]. \qquad (15)$$

Let us discuss some particular cases of the I-functions from which these arose initially, in fractional order models, say in statistical physics, statistical mechanics, stochastics.

Feynman integrals: Rathie introduced the I-function in the goals to cover some important functions of Applied Mathematics that are not included in the H-functions, among them are some Feynman integrals. Inayat-Hussain demonstrated the usefulness of such integrals in derivation of new transformation, summation and reduction formulae for single- and multiple-variable hypergeometric series. His results in evaluating Feynman integrals that arise in perturbation calculations of the equilibrium properties of a magnetic model of phase transitions were based on the function

$$g_1(z) = (2\pi)^{-d} \int_{|p|\le 1} d\mathbf{p} |\mathbf{p}|^{2\eta-d} [\ln(1/|\mathbf{p}|)]^m |\mathbf{1} + z^{1/2}\mathbf{p}|^{-2\gamma}$$

$$= \widetilde{K} \cdot \sum_{k=0}^{\infty} \frac{(\gamma)_k (\gamma-\mu/2)_k \ z^k}{(1+\mu/2)_k (\eta+k)^{1+m} \ k!}.$$

But we can rewrite this in the denotations of the $\widetilde{\Psi}$-function:

$$= \text{Const} \cdot {}_3\widetilde{\Psi}_2 \left[\begin{array}{c} (\gamma,1,1), (\gamma-\mu/2), 1, 1), (\eta, 1, 1+m) \\ (1_\mu/2, 1, 1), (\eta+1, 1, 1+m) \end{array} \middle| z \right], \quad (16)$$

that is as an I-function. Let us recall that m can be a non-integer!

The Gaussian model of phase transitions in equilibrium statistical mechanics. The free energy of such a model on a Bravais lattice in d dimensions was considered by Inayat-Hussain and expressed in terms of a series, where the variable $\varepsilon = \beta_c/\beta - 1$ is a reduced temperature interval and $\beta_c = 2\xi/J$ is the critical temperature:

$$\beta F(d;\varepsilon) = -2^{-2-d}(1+\varepsilon)^{-2}\sum_{k=0}^{\infty}\frac{(1)_k\,[(3/2)_k]^d}{[(2)_k]^{1+d}\,(1+\varepsilon)^{2k}}\,.$$

Inayat-Hussain presented this Gaussian model's free energy by the $\overline{H}$-function:

$$\beta F(d;\varepsilon) = -\frac{(1+\varepsilon)^{-2}}{4\pi^{d/2}}\overline{H}^{1,3}_{3,2}\left[-(1+\varepsilon)^{-2}\,\middle|\,\begin{matrix}(0,1,1),(0,1,1),(-1/2,1,d)\\(0,1,1)(-1,1,1+d)\end{matrix}\right]$$

Having in mind the above series representation, we can write it in terms of the $\widetilde{\Psi}$-function as follows, with NON-integer d:

$$\beta F(d;\varepsilon) = \text{const}\cdot{}_2\widetilde{\Psi}_1\left[\begin{matrix}(1,1,2),(3/2,1,d)\\(2,1,1+d)\end{matrix}\,\middle|\,(1+\varepsilon)^{-2}\right]. \quad (17)$$

Attention: It happens that some other important SF are *NOT* H-functions and ${}_p\Psi_q$-functions, but can be presented in terms of the I- and $\overline{H}$-functions. This was an argument in the initial works by Inayat-Hussain (with title: "... hypergeometric series derivable from Feynman integrals") and by Rathie, and we stuck on some hints that other more popular SF in Maths fall in the scheme of the I-functions, as: the polylogaritm function, the Riemann ζ-function, Mathiew series, etc. Just for example, the polylogarithm function

$$\mathrm{Li}_\alpha(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^\alpha}, \quad |z| < 1, \ \alpha \in \mathbb{C},$$

can be identified as such a function. Namely, a Mellin-Barnes type integral representation gives a $\overline{H}$-function (then, also I-f.):

$$\mathrm{Li}_\alpha(z) = -\frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} \frac{\Gamma^{\alpha+1}(s)\,\Gamma(1-s)}{\Gamma^\alpha(1+s)} (-z)^s \, ds$$

$$= -\overline{H}_{1,2}^{1,1}\left[-z \,\middle|\, \begin{array}{c} (1,1,\alpha+1) \\ (1,1,1),(0,1,\alpha) \end{array} \right], \ \ \alpha > 0, \ \ \mathcal{L} = \{c-i\infty, c+i\infty\}.$$

(All singularities of the Gamma's in numerator are to the left of $s=0$ and to the right of $s=1$, and one can take $c = 1/2$!)

Next example: The generalized Hurwitz-Lerch Zeta function:

$$\Phi_{\lambda,\nu,\mu}^{(\rho,\sigma,\kappa)}(z,\alpha,b) = \sum_{n=0}^{\infty} \frac{(\lambda)_{\rho n}(\mu)_{\sigma n}}{(\nu)_{\kappa n}\, n!} \cdot \frac{z^n}{(n+b)^{\alpha}}, \quad |z| < \rho^*. \tag{18}$$

According to Srivastava-Saxena-Pogány-Saxena (2011), it has the following Mellin-Barnes contour integral representation:

$$\Phi_{\lambda,\nu,\mu}^{(\rho,\sigma,\kappa)}(z,\alpha,b) = \frac{\Gamma(\nu)}{\Gamma(\lambda)\Gamma(\mu)}$$

$$\times \int_{\mathcal{L}} \frac{\Gamma(-s)\Gamma(\lambda+\rho s)\Gamma(\mu+\sigma s)\,\Gamma^{\alpha}(s+b)}{\Gamma(\nu+\kappa s)\,\Gamma^{\alpha}(s+b+1)}\,(-z)^{\alpha}\, ds, \tag{19}$$

for $|\arg(-z)| < \pi$, and path of integration $\mathcal{L} = (c - i\infty, c + i\infty)$ that separates the poles of $\Gamma(-s)$, $\Gamma(\lambda+\rho s)$, $\Gamma(\mu+\sigma s)$, $\Gamma(s+b)$. Then, the relation with the $\overline{H}$-function (and I-f.) can be written:

$$\Phi_{\lambda,\nu,\mu}^{(\rho,\sigma,\kappa)}(z,\alpha,b) = \frac{\Gamma(\nu)}{\Gamma(\lambda)\Gamma(\mu)}$$

$$\times \overline{H}_{3,3}^{1,3}\left[-z \,\middle|\, \begin{matrix} (1-\lambda,\rho,1),(1-\mu,\sigma,1),(1-b,1,\alpha) \\ (0,1),(1-\nu,\kappa,1),(-b,1,\alpha) \end{matrix}\right]. \tag{20}$$

Several important special cases are considered, incl. the **Riemann Zeta function** $\zeta(\alpha) = \sum_0^{\infty} z^n/n^{\alpha}$ (with $z=1,\ b=0,\ \ldots$), etc.

Our hypothesis was, now proved (our papers of 2024), that under suitable conditions on the contour and location of the singularities of the Γ-functions, the Le Roy type functions can also be written in terms of I- and in particular, $\overline{H}$-functions and as generalized Fox-Wright functions ${}_p\widetilde{\Psi}_q$. Namely:

- the original Le Roy function:

$$F^{(\gamma)}(z) = I_{1,2}^{1,1}\left[-z \,\middle|\, \begin{matrix} (0,1,1) \\ (0,1,1),(0,1,\gamma) \end{matrix}\right] = {}_1\widetilde{\Psi}_1\left[\begin{matrix} (1,1,1) \\ (1,1,\gamma \end{matrix} \,\middle|\, z\right];$$

- the M-L type Le Roy function (Gerhold, Garra-Polito,...):

$$F_{\alpha,\beta}^{(\gamma)}(z) = I_{1,2}^{1,1}\left[-z \,\middle|\, \begin{matrix} (0,1,1) \\ (0,1,1),(1-\beta,\alpha,\gamma) \end{matrix}\right] = {}_1\widetilde{\Psi}_1\left[\begin{matrix} (1,1,1) \\ (\beta,\alpha,\gamma \end{matrix} \,\middle|\, z\right];$$

- the Prabhakar type Le Roy function (Paneva-Konovska):

$$F_{\alpha,\beta,\tau}^{(\gamma)}(z) = I_{1,2}^{1,1}\left[-z \,\middle|\, \begin{matrix} (1-\tau,1,1) \\ (0,1,1),(1-\beta,\alpha,\gamma) \end{matrix}\right]$$
$$= \Gamma(\tau)\,{}_1\widetilde{\Psi}_1\left[\begin{matrix} (\tau,1,1) \\ (\beta,\alpha,\gamma \end{matrix} \,\middle|\, z\right];$$

also, the multi-index M-L function of Le Roy type (Rogosin, 2022):

$$F^{(\gamma_i)_1^m}_{(\alpha_i)_1^m,(\beta_i)_1^m}(z) = I^{1,1}_{1,m+1}\left[-z \,\middle|\, \begin{matrix} (0,1,1) \\ (0,1,1),(1-\beta_i,\alpha_i,\gamma_i)_1^m \end{matrix}\right] = \overline{H}^{1,1}_{1,m+1}(-z).$$

Compare: $E_{(\alpha_i)_1^m,(\beta_i)_1^m}(z) = H^{1,1}_{1,m+1}\left[-z \,\middle|\, \begin{matrix} (0,1) \\ (0,1),(1-\beta_i,\alpha_i)_1^m \end{matrix}\right]$, VK, H-f.

Theorem. (2023) Let $\forall \alpha_i, \beta_i, \tau_i > 0, i = 1, \dots, m, \sum_{k=0}^{\infty} \alpha_i \gamma_i > 0$, we have for our multi-index M-L functions of Le Roy type

$$\mathbb{F}_m(z) := \mathbb{F}^{\gamma_i;m}_{\alpha_i,\beta_i,\tau_i}(z) = \sum_{k=0}^{\infty} \frac{(\tau_1)_k \dots (\tau_m)_k}{\prod_{i=1}^{m} \Gamma^{\gamma_i}(\alpha_i k + \beta_i)} \cdot \frac{z^k}{(k!)^m}, \quad \text{that:} \quad (21)$$

$$\mathbb{F}_m(z) := \mathbb{F}^{\gamma_i;\,m}_{\alpha_i;\beta_i;\,\tau_i}(z) = T \cdot {}_m\widetilde{\Psi}_{2m-1}\left[\begin{matrix} (\tau_i,1,1)_1^m \\ (1,1,1)_{(m-1)-\text{times}},(\beta_i,\alpha_i,\gamma_i)_1^m \end{matrix} \,\middle|\, z\right]$$

$$= T \cdot \overline{H}^{1,m}_{m,2m}\left[-z \,\middle|\, \begin{matrix} (1-\tau_i,1,1)_1^m \\ (0,1)_{m-\text{times}},(1-\beta_i,\alpha_i,\gamma_i)_1^m \end{matrix}\right] \quad (22)$$

$$= T \cdot I^{1,m}_{m,2m}\left[-z \,\middle|\, \begin{matrix} (1-\tau_i,1,1)_1^m \\ (0,1,1)_{m-\text{times}},(1-\beta_i,\alpha_i,\gamma_i)_1^m \end{matrix}\right].$$

where $T := 1/(\prod_{i=1}^{m} \Gamma(\tau_i)()$.

Consider now a simple multi-index case with $m = 2$. Pogany considered the problem for a closed-form definite integral expression for the COM–Poisson renormalization constant. He mentioned, as an example only, a special function of the following form that happens to be a generalized Fox-Wright function:

$$F^{\alpha,\beta}_{(p,q;r,s)}(z)=\sum_{k=0}^{\infty}\frac{z^k}{[\Gamma(pk+q)]^{\alpha}[\Gamma(rk+s)]^{\beta}}={}_1\widetilde{\Psi}_2\left[\begin{array}{c}(1,1,1)\\(q,p,\alpha),(s,r,\beta)\end{array}\middle|\,z\right].$$

It happens also that the original Le Roy function $(m = 1, \alpha = \beta = 1)$ plays a practical role in describing other real-world processes. For example, a query has been raised by Kolokoltsov about the simplest case of the Le Roy function with index $\gamma = 1/2$:

$$F^{(1/2)}(z) = \sum_{k=0}^{\infty}\frac{z^k}{(k!)^{1/2}},$$

what kind of, if known, special function is this? Then, in a recent paper (2021), he recognized it and emphasized that this function plays the same role for stochastic equations as the exponential and Mittag-Leffler functions for deterministic equations.

Related “Eigen”-operators for the new SF: Gelfond–Leontiev operators and new operators of FC

In analysis, linear algebra, physics, quantum mechanics, etc., the notions related to the prefix “Eigen” (from the German word “self” or “own”) play important roles. An eigenfunction is a function that, when acted on by an operator, yields a scalar multiple of the function itself. The scalar is called the eigenvalue. Aside from the other analytical properties of the SF, and evaluation of their images under integral transforms and operators of FC, it is an important *but often still open problem* to determine corresponding linear integral L and differential operators D that transform a function f into itself multiplied by a scalar, e.g., $D\,f = \lambda\,f$. For shortness, we call such operators “eigen“-operators for f.

It happens that an useful tool to resolve this (generally open) problem for some classes of SF is the notion of Gelfond–Leontiev operators (G-L operators) for generalized integration and differentiation, introduced by these authors in 1951. Up to now, we have used this theory to propose integral and differential operators for which the ML and multi-ML functions are eigenfunctions.

Definition. (Gelfond-Leontiev, 1951) Let the function $\varphi(\lambda) = \sum_{k=0}^{\infty} \varphi_k \lambda^k$ be an entire function with a growth (order $\rho > 0$ and type $\sigma \neq 0$), such that $\lim_{k\to\infty} k^{\frac{1}{\rho}} \sqrt[k]{|\varphi_k|} = (\sigma e \rho)^{\frac{1}{\rho}}$. Then, for an analytic function f the operation

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \overset{D_\varphi}{\longmapsto} D_\varphi f(z) = \sum_{k=1}^{\infty} a_k \frac{\varphi_{k-1}}{\varphi_k} z^{k-1}, \qquad (23)$$

is called a G-L operator of generalized differentiation with respect to the function $\varphi(\lambda)$. And the corresponding G-L operator of generalized integration can be also introduced:

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \overset{L_\varphi}{\longmapsto} L_\varphi f(z) = \sum_{k=0}^{\infty} a_k \frac{\varphi_{k+1}}{\varphi_k} z^{k+1}. \qquad (24)$$

Evidently, $D_\varphi L_\varphi f(z) = f(z)$. It happens also that the function φ is *eigenfunction* of the G-L operator generated by itself.

The classical diff./integr. are generated by $\varphi(\lambda) = \exp \lambda$; and for the ML function and multi-index ML functions we have constructed corresp. G-L operators, defined by series of the above forms. Then, for these G-L gen. integr. we provided also representations as FC operators with kernels $H^{1,0}_{1,1}$, resp. $H^{m,0}_{m,m}$ (see next).

Now: we construct G-L operators **D** and **L**, generated by the Le Roy type functions - in the case of $F^{(\gamma)_m}_{(\alpha,\beta)_m}$, by means of their coefficients $\varphi_k = 1/\prod_{i=1}^m \Gamma^{\gamma_i}(\alpha_i k+\beta_i)$. For $f(z)=\sum\limits_{k=0}^{\infty} a_k z^k$:

$$\mathbf{D}\, f(z) := D_{mMLR}\, f(z) = \sum_{k=1}^{\infty} a_k z^{k-1} \cdot \prod_{i=1}^{m} \frac{\Gamma^{\gamma_i}(\alpha_i k+\beta_i)}{\Gamma^{\gamma_i}(\alpha_i k+\beta_i-\alpha_i)}, \quad (25)$$

$$\mathbf{L}\, f(z) := L_{mMLR}\, f(z) = \sum_{k=0}^{\infty} a_k z^{k+1} \cdot \prod_{i=1}^{m} \frac{\Gamma^{\gamma_i}(\alpha_i k+\beta_i)}{\Gamma^{\gamma_i}(\alpha_i k+\beta_i+\alpha_i)}. \quad (26)$$

We have proved the following *eigenfunction relation*:

$$\mathbf{D}\, F^{(\gamma)_m}_{(\alpha,\beta)_m}(\lambda z) = \lambda\, F^{(\gamma)_m}_{(\alpha,\beta)_m}(\lambda z), \quad \lambda \neq 0. \quad (27)$$

This result can be interpreted that: the multi-index ML functions of Le Roy type (mMLR) appear as solutions of differential equations of the form (27), and as will be shown below, these are DEs of fractional multi-order $(\alpha_1, ..., \alpha_m)$!

For the G-L integration the corresponding relation is

$$\mathbf{L}\, F^{(\gamma)_m}_{(\alpha,\beta)_m}(\lambda z) = \frac{1}{\lambda}\, F^{(\gamma)_m}_{(\alpha,\beta)_m}(\lambda z) - 1/(\lambda \prod_{i=1}^{m} \Gamma^{\gamma_i}(\beta_i)), \quad \lambda \neq 0. \quad (28)$$

Theorem. (2024) *The G-L integration operator* $\mathbf{L}$*, generated by means of the Le Roy type function* $F_{(\alpha,\beta)_m}^{(\gamma)_m}$ *as in the series* (26)*, can be represented also by means of the integral operator*

$$\mathbb{I}^m f(z) = \mathbf{L}_{mMLR} f(z) = z \int\limits_0^1 I_{m,m}^{m,0} \left[\sigma \left| \begin{array}{c} (\beta_i, \alpha_i, \gamma_i)_1^m \\ (\beta_i - \alpha_i, \alpha_i, \gamma_i)_1^m \end{array} \right. \right] f(z\sigma) d\sigma. \tag{29}$$

This operator can be interpreted as a kind of a generalized fractional integration of multi-order $(\alpha_1, ..., \alpha_m)$*.*

Note that according to the theory of the I-functions, the *singular* kernel in (29) is a well defined $I_{m,m}^{m,0}$-function, analytic in the unit disc $|z| < 1$ that vanishes for $|z| > 1$ (similarly to the behavior of the $H_{m,m}^{m,0}$-function), and under the assumed conditions on the parameters this improper integral is convergent.

Specially for $m = 1$, we can consider this operator as an analogue of the Erdélyi-Kober fractional integral, to be of order $\alpha > 0$:

$$\mathbb{I}^1 f(z) = \mathbf{L}_{MLR} f(z) = z \int_0^1 I_{1,1}^{1,0} \left[\sigma \left| \begin{matrix} (\beta, \alpha, \gamma)_1^m \\ (\beta - \alpha, \alpha, \gamma) \end{matrix} \right. \right] f(z\sigma) d\sigma. \quad (30)$$

Analogy with: Erdélyi-Kober (E-K) fractional integration operator:

$$I_\beta^{\gamma,\delta} f(z) = \frac{1}{\Gamma(\delta)} \int_0^1 (1-t)^{\delta-1} t^\gamma f(zt^{1/\beta})\, dt \quad (31)$$

$$= \int_0^1 H_{1,1}^{1,0} \left[\sigma \left| \begin{matrix} (\gamma + \delta + 1 - \frac{1}{\beta}, \frac{1}{\beta}) \\ (\gamma + 1 - \frac{1}{\beta}, \frac{1}{\beta}) \end{matrix} \right. \right] f(z\sigma), \quad \text{of order } \delta > 0.$$

The E-K integral of "classical" FC, reduces to the R-L fractional integral for $\gamma = 0, \beta = 1$, namely: $I_1^{0,\delta} f(z) = z^{-\delta} I^\delta f(z)$. And the corresponding Erdélyi-Kober (E-K) fractional derivative, as introduced in Kiryakova (1994, Ch.2), has the form

$$D_\beta^{\gamma,\delta} f(z) = D_\eta\, I_\beta^{\gamma+\delta,\eta-\delta} f(z), \;\text{ with }\; n-1 < \delta \leq \eta,\, \eta \in \mathbb{N},$$

$$\text{and } D_\eta := \prod_{j=1}^{\eta} \left(\frac{1}{\beta} z \frac{d}{dz} + \gamma + j\right), \text{ a polynomial } P\left(\frac{d}{dz}\right).$$

Generalized FC (Kiryakova, 1994): Let $m \geq 1$ be an integer; $\delta_i \geq 0$, $\gamma_i \in \mathbb{R}$, $\beta_i > 0$, $i = 1, \ldots, m$. In analogy with the definition of the E-K integral, consider the sets of parameters: $\delta = (\delta_1 \geq 0, ..., \delta_m \geq 0)$ as a multi-order of fractional integration, $\gamma = (\gamma_1, ..., \gamma_m)$ as a multi-weight, and additional multi-parameter $\beta = (\beta_1 > 0, ..., \beta_m > 0)$. The integral operator defined as follows:

$$I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} f(z) = \int_0^1 H^{m,0}_{m,m} \left[t \,\middle|\, \begin{array}{c} (\gamma_i + \delta_i + 1 - \frac{1}{\beta_i}, \frac{1}{\beta_i})_1^m \\ (\gamma_i + 1 - \frac{1}{\beta_i}, \frac{1}{\beta_i})_1^m \end{array} \right] f(zt)dt, \tag{32}$$

if $\sum_{i=1}^{m} \delta_i > 0$; and $I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} f(z) = f(z)$ for $\forall \delta_i = 0$, is called a generalized (m-tuple) fractional integral of multi-order $(\delta_1 \geq 0, \ldots, \delta_m \geq 0)$.

The generalized fractional derivatives $D^{(\gamma_k),(\delta_k)}_{(\beta_k),m}$, corresponding to (32), of multi-order $(\delta_1, ..., \delta_m)$ are also introduced, in a way similar to that for the R-L D^{δ} and E-K $D^{\gamma,\delta}_{\beta}$ FDs, but with more complicated differ-integral representations.

For $m = 1$, the g.f.i. (32) is the "classical" E-K integral (31).

The main feature of the operators (32) defined by means of single integrals involving H-functions (or Meijer's G-functions in the simpler case of equal $\beta_i = \beta > 0, i = 1, ..., m$) is that they can be equivalently represented by means of commutative compositions of classical E-K integrals ($m = 1$):

$$I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} f(z) = \left[\prod_{i=1}^{m} I_{\beta_i}^{\gamma_i,\delta_i} \right] f(z)$$

$$= \int_0^1 \ldots \int_0^1 \left[\prod_{i=1}^{m} \frac{(1-t_i)^{\delta_i - 1} t_i^{\gamma_i}}{\Gamma(\delta_i)} \right] \; f\left(z t_1^{\frac{1}{\beta_1}} \ldots \sigma_m^{\frac{1}{\beta_m}} \right) dt_1 \ldots dt_m \,, \tag{33}$$

without special functions involved in kernel.

The frequent appearance of compositions like (33) in problems related to applications, combined with the simple but effective tools of the theory of the kernel SFs (H- and G-functions) in definition (32), explains the wide usage of the operators of the GFC.

Return now to the G-L operators generated by the Le Roy type functions $F^{(\gamma)_m}_{(\alpha,\beta)_m}(z)$, and their integral representations (29) as in the previous theorem. In the case $m = 1$ the operators $\mathbb{I}^1$ can be considered as analogues of the Erdélyi-Kober operators ! And, it happens that the following composition/decomposition property for $\mathbb{I}^m$ and $\mathbb{I}^1$, is analogous to the mentioned one for the GFC, where the gen. fr. integrals with $H^{m,0}_{m,m}$ were represented also as commutable compositions of m single E-K fractional integrals.

Theorem. (2024) *For entire functions $f(z)$,*

$$\mathbb{I}^m f(z) = \left[\prod_{i=1}^{m} \mathbb{I}^1_i\right] f(z) = \mathbb{I}^1_m \left\{\mathbb{I}^1_{m-1} \cdots \left[\mathbb{I}^1_1\right]\right\} f(z). \qquad (34)$$

The above composition is commutable. The analogues of the E-K fractional integrals of order $\alpha_i > 0$ have the form

$$\mathbb{I}^1_i f(z) = z \int_0^1 I^{1,0}_{1,1} \left[\sigma \,\middle|\, \begin{matrix} (\beta_i, \alpha_i, \gamma_i) \\ (\beta_i - \alpha_i, \alpha_i, \gamma_i) \end{matrix} \right] f(z\sigma) d\sigma. \qquad (35)$$

We may consider the operators in (29) and (34), also with more general parameters, as new kind of generalized fractional integrals, with semigroup and other operational properties, typical for the FC.

Say for $\underline{m=1}$, the more general form of such integral operators (as analogues of E-K integrals), but depending on 4 (instead of 3) parameters, can be considered in the form:

$$\mathbb{I}_{\beta,\gamma}^{\mu,\alpha} f(z) = \int_0^1 I_{1,1}^{1,0} \left[\sigma \left| \begin{array}{c} (\mu+\alpha-\beta,\beta,\gamma) \\ (\mu-\beta,\beta,\gamma) \end{array} \right. \right] f(z\sigma) d\sigma,$$

for order $\alpha > 0$; and for $\alpha = 0$: $\mathbb{I}_{\beta,\gamma}^{\mu,0} f(z) := f(z)$. It is easy to check that the semi-group property is then satisfied OK:

$$\mathbb{I}_{\beta,\gamma}^{\mu+\alpha_1,\alpha_2} \, \mathbb{I}_{\beta,\gamma}^{\mu,\alpha_1} = \mathbb{I}_{\beta,\gamma}^{\mu,\alpha_1+\alpha_2}, \ \alpha_1 > 0, \alpha_2 > 0,$$

in the form very similar to that for the E-K fractional integrals.

The case of arbitrary $m \geq 1$, for GFC, can be also considered for:

$${}^{m}\mathbb{I}_{(\beta_i),(\gamma_i)}^{(\mu_i),(\alpha_i)} f(z) = \int_0^1 I_{m,m}^{m,0} \left[\sigma \left| \begin{array}{c} (\mu_i+\alpha_i-\beta_i,\beta_i,\gamma_i)_1^m \\ (\mu_i-\beta_i,\beta_i,\gamma_i)_1^m \end{array} \right. \right] f(z\sigma) d\sigma.$$

A list of OPEN problems can been also discussed!