Metric embeddings of Laakso graphs into Banach spaces

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Super-reflexive Banach spaces

Definition

Let X_0, Y_0 be n-dimensional normed spaces. The Banach-Mazur distance from X_0 to Y_0 is defined by

$$d_{BM}(X_0, Y_0) = \inf\{\|T\| \cdot \|T^{-1}\| \colon T \colon X_0 \to Y_0\}.$$

Definition

Let X and Y be infinite-dimensional Banach spaces. Y is finitely representable in X if $\forall \epsilon > 0$ and $\forall$ finite-dimensional subspaces Y_0 of Y $\exists$ a finite-dimensional subspace X_0 of X with

$$d_{BM}(X_0, Y_0) \leqslant 1 + \epsilon.$$

Definition (James, 1972)

X is super-reflexive if

$$Y \text{ is finitely representable in } X \Rightarrow Y \text{ is reflexive (i.e., } Y = Y^{**}).$$

Remark

- super-reflexive $\Rightarrow$ reflexive
- ℓ_p and $L_p[0,1]$ are super-reflexive $\Leftrightarrow 1 < p < \infty$
- $(\sum_{n \geqslant 1} \ell_1^n)_2$ is reflexive but not super-reflexive

Characterizations of super-reflexive spaces

Theorem (Enflo, 1972)

X is super-reflexive if and only if X is isomorphic to a uniformly convex Banach space.

Theorem (James-Schaffer, 1972, Schaffer-Sundaresan, 1970)

X is super-reflexive if and only if X is J-convex:
$\exists m \geqslant 2$, $\varepsilon > 0$ such that $\forall e_1, \ldots, e_m$, $\|e_i\| \leqslant 1$,

$$\min_{1 \leqslant j \leqslant m} \|e_1 + \cdots + e_j - e_{j+1} - \cdots - e_m\| < m - \varepsilon. \qquad (1)$$

Bilipschitz embeddings of metric spaces

Definition

A metric space M bilipschitz embeds in a Banach space X with distortion D if $\exists\ f\colon M \to X$ s.t.

$$\frac{1}{D}\rho(x,y) \leqslant \|f(x) - f(y)\| \leqslant \rho(x,y) \qquad (x,y \in M).$$

Characterization of super-reflexivity: Binary trees

Definition

For $n \geqslant 1$, the binary tree $B_n := \{\emptyset\} \cup_{i=1}^{n} \{0,1\}^i$ equipped with the shortest path metric.

Theorem (Bourgain, 1986)

X is not superreflexive $\Leftrightarrow \exists D \geqslant 1$ and maps $f_n \colon B_n \to X$ s.t.

$$\frac{d(s,t)}{D} \leqslant \|f_n(s) - f_n(t)\| \leqslant d(s,t),$$

i.e., B_n bilipschitz embeds into X with uniform distortion.

Diamond graphs

- The diamond graphs D_n are defined recursively:
- D_0 is a single edge.
- D_n is obtained from D_{n-1} by replacing each edge by a ‘diamond’.
- Equip D_n with the shortest path metric.

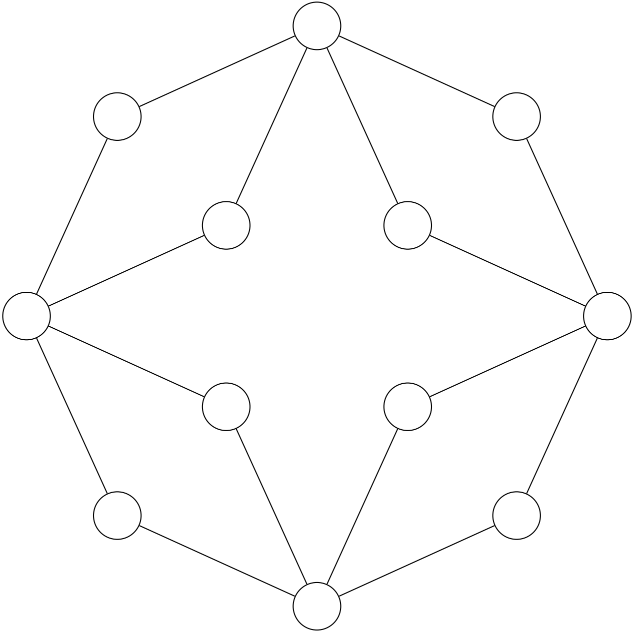

Figure: Diamond D_2.

Laakso graphs

- The Laakso graphs $\mathcal{L}_n$ are defined recursively:
- $\mathcal{L}_0$ is a single edge.
- $\mathcal{L}_n$ is obtained from $\mathcal{L}_{n-1}$ by replacing each edge by a copy of $\mathcal{L}_1$
- Equip $\mathcal{L}_n$ with the shortest path metric.

Theorem (Ostrovska-Ostrovskii, 2017)

Laakso graphs do not uniformly bilipschitz embed into diamond graphs.

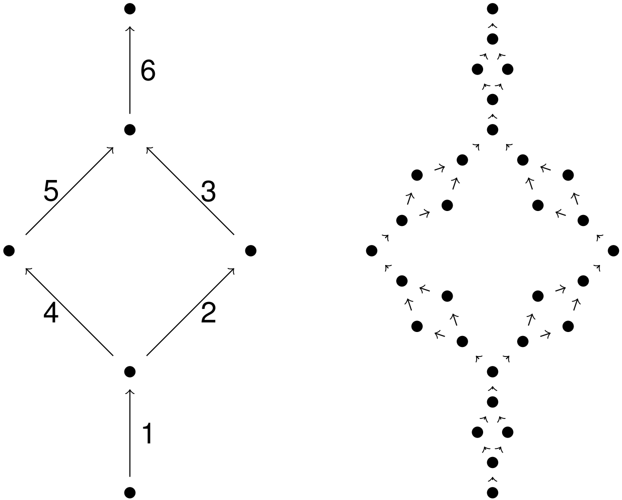

Figure: The Laakso graphs $\mathcal{L}_1$ and $\mathcal{L}_2$

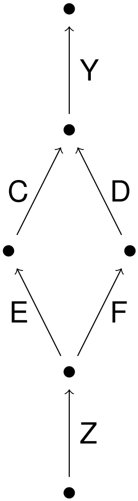

Figure: The Laakso graph $\mathcal{L}_n$

Here C, D, E, F, Y, Z are copies of $\mathcal{L}_{n-1}$.

Characterization of super-reflexivity: diamond and Laakso graphs

Theorem (Johnson-Schechtman, 2009)

Let X be a Banach space. Then X is not superreflexive $\Leftrightarrow \exists D \geqslant 1$ and maps $f_n \colon D_n \to X$ or $f_n \colon \mathcal{L}_n \to X$ such that

$$\frac{d(s,t)}{D} \leqslant \|f_n(s) - f_n(t)\| \leqslant d(s,t),$$

i.e., D_n and $\mathcal{L}_n$ bilipschitz embed into X with uniform distortion.

Further graph characterizations

Suppose X is not super-reflexive. Let $\varepsilon > 0$

Theorem (Ostrovskii-Randrianantoanina, 2017)

The k-branching diamond $D_{n,k}$ and Laakso $\mathcal{L}_{n,k}$ graphs bilipschitz embed into X with uniform distortion $8 + \varepsilon$.

Theorem (Swift, 2018)

Bundle graphs generated by a finitely-branching bundle graph bilipschitz embed with distortion indepenedent of the branching number. Parasol graphs embed with distortion $8 + \varepsilon$.

Low distortion embeddings of diamond graphs D_n

Suppose X is not super-reflexive. Let $\varepsilon > 0$

Theorem (folklore)

The binary trees B_n bilipschitz embed into X with uniform distortion $1 + \varepsilon$ (B_n embeds almost isometrically into X).

Theorem (Pisier, 2016)

D_n bilipschitz embeds into X with uniform distortion $2 + \varepsilon$.

Theorem (Lee and Rhagavendra, 2010)

D_n bilipschitz embeds into $L_1[0, 1]$ with uniform distortion $4/3$.

Low distortion embeddings of Laakso graphs $\mathcal{L}_n$

Here X is not super-reflexive and $\varepsilon > 0$.

Theorem (DKS, 2022)

$\mathcal{L}_n$ bilipschitz embed into X with distortion $2 + \varepsilon$.

Theorem (DKS, 2022)

$\mathcal{L}_n$ bilipschitz embed into $L_1[0, 1]$ with distortion $4/3$.

Lower bounds on distortion

Theorem (DKS)

The diamond graph D_2 does not embed into $L_1[0,1]$ with distortion less than $5/4$.

Remark

$\exists$ simple embedding of D_2 into $L_1[0,1]$ with distortion $4/3$ which may be optimal, but we don't have a proof.

Theorem (DKS, 2022)

The Laakso graph $\mathcal{L}_2$ does not embed into $L_1[0,1]$ with distortion less than $9/8$.

Sketch of the proofs of the results

Theorem

Suppose X is not super-reflexive. $\forall\, \varepsilon > 0$ *and* $\forall\, n \geqslant 1$, $\exists$ $f_n \colon \mathcal{L}_n \to X$ *s.t.* $\forall\, a, b \in \mathcal{L}_n$,

$$\frac{1}{2} d(a,b) - \varepsilon \leqslant \|f_n(a) - f_n(b)\| \leqslant d(a,b). \tag{2}$$

- Since X is not J-convex, $\exists\, (e_i^n)_{i=1}^{4^n}$ s.t. $\|e_i\| \leqslant 1$ and

$$\min_{1 \leqslant j \leqslant 4^n} \|e_1 + \cdots + e_j - e_{j+1} - \cdots - e_{4^n}\| \geqslant 4^n - \varepsilon.$$

- f_n is of form

$$f_n(a) = \sum_{i=1}^{4^n} (e_i^n)^*(f_n(a)) e_i^n, \tag{3}$$

where $(e_i^n)^*(f_n(a)) \in \{0, 1\}$

- The proof is inductive:

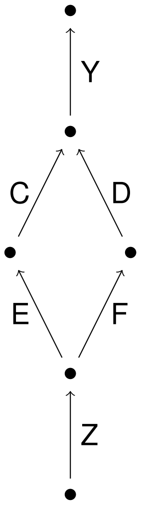

Figure: The Laakso graph $\mathcal{L}_n$

Inductive definition

- Let $\rho\colon \mathcal{L}_{n-1} \to X$ be a 'copy' of f_{n-1} with $(e_i^{n-1})_{i=1}^{4^{n-1}}$ replaced by $(e_i^n)_{i=1}^{4^{n-1}}$. Formally,

$$\rho(a) = \sum_{i=1}^{4^{n-1}} (e_i^{n-1})^*(f_{n-1}(a))e_i^n.$$

- Let $\theta\colon \mathcal{L}_{n-1} \to X$ be a copy of f_{n-1} with $(e_i^{n-1})_{i=1}^{4^{n-1}}$ replaced by $(e_i^n)_{i=4^{n-1}+1}^{2\cdot 4^{n-1}}$.
- Let $\phi\colon \mathcal{L}_{n-1} \to X$ be a copy of f_{n-1} with $(e_i^{n-1})_{i=1}^{4^{n-1}}$ replaced by $(e_i^n)_{i=2\cdot 4^{n-1}+1}^{3\cdot 4^{n-1}}$.
- Let $\sigma\colon \mathcal{L}_{n-1} \to X$ be a copy of f_{n-1} with $(e_i^{n-1})_{i=1}^{4^{n-1}}$ replaced by $(e_i^n)_{i=3\cdot 4^{n-1}+1}^{4^n}$.

- Now we define $f_n \colon \mathcal{L}_n \to X$ as follows:

$$f_n(a) = \begin{cases} \rho(\overline{a}), & a \in Y \\ \sum_{i=1}^{4^{n-1}} e_i^n + \theta(\overline{a}), & a \in C \\ \sum_{i=1}^{4^{n-1}} e_i^n + \phi(\overline{a}), & a \in D \\ \sum_{i=1}^{2\cdot 4^{n-1}} e_i^n + \phi(\overline{a}), & a \in E \\ \sum_{i=1}^{4^{n-1}} e_i^n + \sum_{i=2\cdot 4^{n-1}+1}^{3\cdot 4^{n-1}} e_i^n + \theta(\overline{a}), & a \in F \\ \sum_{i=1}^{3\cdot 4^{n-1}} e_i^n + \sigma(\overline{a}), & a \in Z. \end{cases}$$

- Check $\|(f_n(a) - f_n(b)\|$ case by case, e.g. $a \in D, b \in E$ (Case 4 in the paper).

Lower estimate for $\|f_n(a) - f_n(b)\|$

Let $(e_i)_{i=1}^m$ satisfy $\|e_i\| \leqslant 1$ and

$$\min_{1\leqslant j\leqslant m} \|e_1 + \cdots + e_j - e_{j+1} - \cdots - e_m\| \geqslant m - \varepsilon.$$

Lemma

$\max A < \min B \Rightarrow$

$$\|\sum_{i\in A} e_i - \sum_{i\in B} e_i\| \geqslant |A| + |B| - \varepsilon.$$

Lemma

$\max A < \min B$ *or* $\max B < \min A \Rightarrow$

$$\|\sum_{i\in A} \varepsilon_i e_i + \sum_{i\in B} e_i\| \geqslant |B| - \varepsilon.$$

for all choices of signs $\varepsilon_i = \pm 1$.

Bilipschitz embedding into $L_1[0,1]$.

Theorem

$\forall\ n \geqslant 1,\ \exists\ f_n\colon \mathcal{L}_n \to L_1[0,1]$ *s.t.* $\forall\ a, b \in \mathcal{L}_n$,

$$\frac{3}{4} d(a,b) \leqslant \|f_n(a) - f_n(b)\|_1 \leqslant d(a,b).$$

Proof.

Similar but uses independent sets to improve $1/2$ to $3/4$. □

Lower bounds on distortion

Theorem

Let $f\colon \mathcal{L}_2 \to L_1[0,1]$ *satisfy*

$$d(a,b) \leqslant \|f(a) - f(b)\|_1 \leqslant cd(a,b).$$

Then $c \geqslant 9/8$*.*

Theorem

Let $f\colon D_2 \to L_1[0,1]$ *satisfy*

$$d(a,b) \leqslant \|f(a) - f(b)\|_1 \leqslant cd(a,b).$$

Then $c \geqslant 5/4$*.*

Hypermetric and negative type inequalities

Theorem B (Deza-Laurent, 1997)

Let (M, ρ) be a finite metric space that embeds isometrically into $L_1[0, 1]$.
$\forall\, k_i \in \mathbb{Z}$ ($1 \leqslant i \leqslant n$) s.t. $\sum_{i=1}^{n} k_i = 0$ (negative type inequalities) or $\sum_{i=1}^{n} k_i = 1$ (hypermetric inequalities),

$$\sum_{1 \leqslant i < j \leqslant n} k_i k_j \rho(x_i, x_j) \leqslant 0,$$

where $x_1, \ldots, x_n$ are the distinct elements of M.

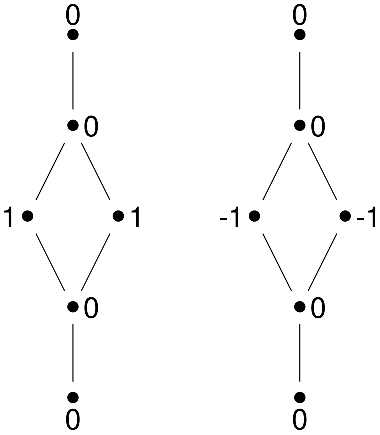

Figure: Weights P (left) and N (right) for $\mathcal{L}_1$

$P \to \{C, F\}$, $N \to \{D, E\}$, zero $\to \{Y, Z\}$ copies of $\mathcal{L}_1$ in $\mathcal{L}_2$.

- $\sum_{i=1}^{30} k_i = 0 \Rightarrow$ negative type inequailty
- $$\begin{aligned} 72 &= \sum_{i<j, k_i k_j > 0} k_i k_j d(x_i, x_j) \\ &\leqslant \sum_{i<j, k_i k_j > 0} k_i k_j \|f(x_i) - f(x_j)\|_1 \\ &\leqslant \sum_{i<j, k_i k_j < 0} |k_i k_j| \|f(x_i) - f(x_j)\|_1 \\ &\leqslant c \sum_{i<j, k_i k_j < 0} |k_i k_j| d(x_i, x_j) \\ &= 64c. \end{aligned}$$
- So $c \geqslant 9/8$.