Unimodality Preservation by Ratios of Functional Series and Integral Transforms

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Biernaki–Krzyż and beyond

Lemma (Biernaki–Krzyż, 1955)

Suppose two real power series $A(x) = \sum_{k=0}^{\infty} a_k x^k$ and $B(x) = \sum_{k=0}^{\infty} b_k x^k$ both converge in $(-r, r)$, $0 < r \leq \infty$ and $b_k > 0$ for all k. If the non-constant sequence $\{a_k/b_k\}$ is increasing (resp. decreasing) for all k, then the function $x \to A(x)/B(x)$ is strictly increasing (resp. decreasing) on $(0, r)$. Briefly: monotonicity is inherited by power series ratios from that of coefficient ratios.

Polynomial version: the above also holds when $a_k = b_k = 0$ for all $k > n$, so that $\{a_k/b_k\}$ is a finite sequence; it has been rediscovered several times, including Heikkala, Vamanamurthy and Vourinen (2009) and Karp and Sitnik (2010), Geno Nikolov (recently). Biernaki–Krzyż lemma has been extended in three direction:

- more complicated monotonicity pattern of a_k/b_k
- replacing x^k with more general functional sequence $\phi_k(x)$
- replacing series by integral (Laplace transform or more general)

F. Belunce, E.-M. Ortega and J.M.Ruiz (Murcia, Alicante) in their 2007 paper in *Insurance: mathematics & economics* stated the following:

Lemma (Belunce, Ortega and Ruiz, 2007)

Suppose two real power series $A(x)=\sum_{k=0}^{\infty}a_kx^k$ and $B(x)=\sum_{k=0}^{\infty}b_kx^k$ both converge for all finite x. If the sequence $\{a_k/b_k\}$ is increasing for $0\le k\le k_0$ and decreasing for $k\ge k_0$ with $k_0>0$, then the function $x\to A(x)/B(x)$ is increasing on $(0,x_0)$ and decreasing on (x_0,∞) for some $x_0>0$. Briefly: unimodality is inherited by power series ratios from that of coefficient ratios.

They gave an indication for the proof which while is incorrect formally points to the right direction. Our investigation is a continuation of their line of thought (details below).

Yang, Chu and Wang

The result of Belunce, Ortega and Ruiz was quoted and used by Baricz, Vesti and Vourinen (2011) and Simić and Vourinen (2012) with infinite radius of convergence replaced by a finite positive value. In 2015 Zh.-H. Yang, Y.-M. Chu and M.-K. Wang showed that this replacement may sometimes lead to wrong conclusions. They introduced $H_{A,B}(x) = B(x)A'(x)/B'(x) - A(x) = B^2(x)(A(x)/B(x))'/B'(x)$, satisfying $\mathrm{sign}(A/B)' = \mathrm{sign}(B')\mathrm{sign}(H_{A,B})$ and proved

Theorem (Yang, Chu and Wang, J. Math. Anal. and Appl., 2015)

Suppose two real power series $A(x) = \sum_{k=0}^{\infty} a_k x^k$ and $B(x) = \sum_{k=0}^{\infty} b_k x^k$ both converge in $(-r, r)$, $0 < r \leq \infty$ and $b_k > 0$ for all k. If the sequence $\{a_k/b_k\}$ is increasing for $0 \leq k \leq k_0$ and decreasing for $k \geq k_0$ with $k_0 > 0$, then the function $x \to A(x)/B(x)$ is increasing on $(0, x_0)$ and decreasing on (x_0, ∞) for some $x_0 > 0$ if $H_{A,B}(r^-) < 0$. If, on the other hand, $H_{A,B}(r^-) \geq 0$, then it is strictly increasing on $(0, r)$. Furthermore, if $r = \infty$, we have $H_{A,B}(+\infty) < 0$.

Extension to general series

Theorem (Koumandos – Pedersen, Math.Scand., 2009)

Suppose $a_k, b_k > 0$ and two real functional series

$$A(x) = \sum\nolimits_{k=0}^{\infty} a_k u_k(x) \quad \text{and} \quad B(x) = \sum\nolimits_{k=0}^{\infty} b_k u_k(x)$$

both converge absolutely and uniformly on compact subsets of $[0, \infty)$ and so do their term-wise derivatives. Then:

- If the logarithmic derivatives $\{u_k'(x)/u_k(x)\}_{k=0}^{\infty}$ form an increasing sequence of functions and if $\{a_k/b_k\}$ decreases (resp. increases) then $x \to A(x)/B(x)$ decreases (resp. increases) for $x \geq 0$.
- If the logarithmic derivatives $\{u_k'(x)/u_k(x)\}_{k=0}^{\infty}$ form a decreasing sequence of functions and if $\{a_k/b_k\}$ decreases (resp. increases) then $x \to A(x)/B(x)$ increases (resp. decreases) for $x \geq 0$.

Independent alternative proof: G.J.O Jameson, Monotonic ratios of functions, Mathematical Gazette, **105**:562, 2021, 129–134.

Extension to general series

Theorem (Mao – Tian, AMS Proceedings, 2024)

Suppose $u_k(x) \in C_2[0, r)$, $0 < r \leq \infty$, and

- $u_0 = \mathrm{const} > 0$, $u_k(x) > 0$, $u_k'(x) \geq 0$;
- $\lim_{x\to 0^+} u_k(x) = \lim_{x\to 0^+} u_k'(x)/u_1'(x) = 0$ (second equality for $k \geq 2$);
- $\{u_k''(x)/u_k'(x)\}_{k=0}^{\infty}$ form an increasing sequence of functions for each $x \in (0, r)$.

If the sequence $\{a_k/b_k\}$ is increasing for $0 \leq k \leq k_0$ and decreasing for $k \geq k_0$ with $k_0 > 0$, then the function $x \to A(x)/B(x)$ is increasing on $(0, x_0)$ and decreasing on (x_0, ∞) for some $x_0 > 0$ if $H_{A,B}(r^-) < 0$. If, on the other hand, $H_{A,B}(r^-) \geq 0$, then it is strictly increasing on $(0, r)$.

Similar result holds if $k \to u_k'(x) \leq 0, \{u_k''(x)/u_k'(x)\}_{k=0}^{\infty}$ is decreasing, and the boundary conditions are given at $x = r^-$. It this case the monotonicity pattern of $\{a_k/b_k\}$ is reversed by $x \to A(x)/B(x)$.

Further extensions

- Yang, Qian, Chu, Zhang, , J. Inequal. Appl., 2017 – monotonicity for ratios of the type $x \to (P(x) + A(x))/(P(x) + B(x))$, where $A(x)$ and $B(x)$ are power series
- Yang – Tian, J. Math. Anal. and Appl., 2019 – unimodality for ratios of Laplace transforms
- Qi, Comptes Rendus Mathématique, 2022 – monotonicity for ratio of general integral transforms
- Mao, Du and Tian, arXiv:2312.10252v1, December 2023 – – monotonicity for ratios of the type $x \to (P(x) + A(x))/(P(x) + B(x))$, where $A(x)$ and $B(x)$ integral transforms on time scales
- Mao – Tian, arXiv:2404.18168v1 (April, 2024) – ratio of general series, monotonicity changes twice

However! The main theorem of Mao–Tian is not applicable (at least directly) to the functional sequences e^{λ_k}, $(x)_k$ (rising factorial), $1/(x)_k$, $(a+x)_k/(b+x)_k$, $(q^x;q)_k$, $1/(q^x;q)_k$, etc.

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An elementary observation

We will call a real continuous function g defined on a real interval I unimodal if g is either monotonic on I or changes monotonicity direction once. Four possibilities: increasing, decreasing, first increasing then decreasing, or first decreasing then increasing (constant maximal/minimal value on some sub-interval is also allowed).

Master lemma

Let $f : I \to \mathbb{R}$ be a given continuous function on a convex set $I \subseteq \mathbb{R}$. The function f is unimodal if and only if for every $\lambda \in \mathbb{R}$ the function $f_\lambda(x) = f(x) - \lambda$ has no more than 2 sign changes on I, and for all λ such that $f_\lambda(x)$ has exactly 2 sign changes, the sign pattern of the function $f_\lambda(x)$ remains fixed. In a similar fashion, a real sequence $(d_k)_{k=0}^{\infty}$ is a unimodal if and only if for every $\lambda \in \mathbb{R}$ the sequence $(d_k - \lambda)_{k=0}^{\infty}$ has no more than 2 sign changes and for all λ such that it has exactly 2 sign changes, the sign pattern of the sequence $(d_k - \lambda)_{k=0}^{\infty}$ remains fixed.

Sign-regularity and total positivity

Let X, Y be linearly ordered sets, in this talk - some subsets of $\mathbb{R}$. A function (kernel) $K : X \times Y \to \mathbb{R}$ is called sign regular of order $r \in \mathbb{N}$ ($K \in SR_r$) if there is a sequence of signs $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_r$, each $+1$ or -1 such that

$$\varepsilon_m \begin{vmatrix} K(x_1, y_1) & K(x_1, y_2) & \dots & K(x_1, y_m) \\ K(x_2, y_2) & K(x_2, y_2) & \dots & K(x_2, y_m) \\ \vdots & \vdots & \vdots & \ddots \\ K(x_m, y_1) & K(x_m, y_2) & \dots & K(x_m, y_m) \end{vmatrix} \geq 0$$

for all $m = 1, \dots, r$ and any points $x_1 < x_2 < \dots < x_m$, $y_1 < y_2 < \dots < y_m$, $x_i \in X$, $y_j \in Y$. If ≥ 0 above is replaced by > 0 the kernel K is called *strictly sign regular* of order r, $K \in SSR_r$. If $\varepsilon_1 = \dots = \varepsilon_r = 1$, the kernel is *totally positive* of order r, $K \in TP_r$ or *strictly totally positive* ($K \in STP_r$) if all determinants above are strictly positive. If $K \in SR_r$ ($K \in TP_r$) for all $r \in \mathbb{N}$ we write $K \in SR$ (or $K \in SSR$ or $K \in TP$ or $K \in STP$).

Variation diminishing property

Denote by $S^-(f(x))_{x\in I}$ the number of sign changes of a continuous function $f(x)$ on an interval I ignoring zeros (similarly, $S^-(\lambda_n)_{n=0}^\infty$ for a real sequence $(\lambda_n)_{n=0}^\infty$).

Theorem: Motzkin–Schoenberg–Gantmacher–Krein–Karlin

Suppose for a sequence of real continuous functions $(\varphi_n)_{n=0}^\infty$ defined on an interval I, the kernel $K(n,x)=\varphi_n(x)\in SR_r$ on $\mathbb{N}_0\times I$. Then for every sequence of reals $(\lambda_n)_{n=0}^\infty$ having not more that $r-1$ sign changes and such that the series $f(x)=\sum_{n=0}^\infty\lambda_n\varphi_n(x)$ converges uniformly on all compact subsets of I we have

$$S^-(f(x))_{x\in I}\le S^-(\lambda_n)_{n=0}^\infty.$$

Moreover, if $S^-(f(x))_{x\in I}=S^-(\lambda_n)_{n=0}^\infty=k\le r-1$, then the sign patterns of $f(x)$ and $(\lambda_n)_{n=0}^\infty$ coincide if $\varepsilon_k\varepsilon_{k+1}=1$ or are inverse to each other if $\varepsilon_k\varepsilon_{k+1}=-1$. Here $\varepsilon_0=1$ and $\varepsilon_m\det(\phi_{n_i}(x_j))_{i,j=1}^m\ge 0$ for any finite sequences $n_1<n_2\cdots<n_m$, $x_1<\cdots<x_m$ with $m\le r$.

Main theorem for series

Suppose for a sequence of real continuous functions $(\varphi_n)_{n=0}^{\infty}$ defined on a convex interval $I \subseteq \mathbb{R}$, the kernel $K(n,x) = \varphi_n(x) \in SR_3$ on $\mathbb{N}_0 \times I$. Suppose both series in the following definition converge uniformly on compact subsets of I:

$$F(x) = \frac{\sum_{k=0}^{\infty} a_k \varphi_k(x)}{\sum_{k=0}^{\infty} b_k \varphi_k(x)},$$

where $a_k \in \mathbb{R}$, $b_k > 0$, $k = 0, 1, 2, \ldots$. If the sequence of quotients $\left\{\frac{a_k}{b_k}\right\}_{k=0}^{\infty}$ is a unimodal sequence, then the function $F(x)$ is a unimodal function of x. Moreover, if $F(x)$ is not monotonic, then it inherits the monotonicity pattern of the sequence $\left\{\frac{a_k}{b_k}\right\}_{k=0}^{\infty}$ if $\varepsilon_2\varepsilon_3 > 0$ or reverses it if $\varepsilon_2\varepsilon_3 < 0$, where for any $k_1 < k_2 < k_3$ and $x_1 < x_2 < x_3$:

$$\varepsilon_2 \begin{vmatrix} \varphi_{k_1}(x_1) & \varphi_{k_2}(x_1) \\ \varphi_{k_1}(x_2) & \varphi_{k_2}(x_2) \end{vmatrix} \geq 0, \quad \varepsilon_3 \begin{vmatrix} \varphi_{k_1}(x_1) & \varphi_{k_2}(x_1) & \varphi_{k_3}(x_1) \\ \varphi_{k_1}(x_2) & \varphi_{k_2}(x_2) & \varphi_{k_3}(x_2) \\ \varphi_{k_1}(x_3) & \varphi_{k_2}(x_3) & \varphi_{k_3}(x_3) \end{vmatrix} \geq 0.$$

Main theorem for integral transforms

Suppose $I, J \subseteq \mathbb{R}$ are convex intervals. Let $K(x,t)$ be a SR_3 kernel on $I \times J \to \mathbb{R}$ and $w : J \to (0,\infty)$ be a positive weight. Suppose both integrals in the following definition converge uniformly on compact subsets of I:

$$F(x) = \frac{\int_J K(x,t)A(t)w(t)dt}{\int_J K(x,t)B(t)w(t)dt},$$

where $A : J \to \mathbb{R}$ and $B : J \to (0,\infty)$. If the function $t \to A(t)/B(t)$ is unimodal, then the function $x \to F(x)$ is unimodal. Moreover, if $F(x)$ is not monotonic, then it inherits the monotonicity pattern of $t \to A(t)/B(t)$ if $\varepsilon_2\varepsilon_3 > 0$ or reverses it if $\varepsilon_2\varepsilon_3 < 0$,, where for any $t_1 < t_2 < t_3$ and $x_1 < x_2 < x_3$:

$$\varepsilon_2 \begin{vmatrix} K(x_1,t_1) & K(x_1,t_2) \\ K(x_2,t_1) & K(x_2,t_2) \end{vmatrix} \geq 0, \quad \varepsilon_3 \begin{vmatrix} K(x_1,t_1) & K(x_1,t_2) & K(x_1,t_3) \\ K(x_1,t_1) & K(x_2,t_2) & K(x_2,t_3) \\ K(x_3,t_1) & K(x_3,t_2) & K(x_3,t_3) \end{vmatrix} \geq 0.$$

Remarks on the main theorem

Remark 1. All possibilities present in the formulation of the theorem can be realized: $F(x)$ maybe increasing, decreasing or unimodal on I in the direction specified by the theorem. To verify which of the possibilities is realized for a given sequences a_k, b_k, it suffices to verify the sign of the derivatives at the endpoints of I: $F'(a+)$ and/or $F'(b-)$, where $a = \inf I$, $b = \sup I$.

Remark 2. If all functions $x \to \varphi_n(x)$ are $(r-1)-$times continuously differentiable, then a sufficient condition for $K(n,x) = \varphi_n(x) \in SR_r$ is the following: there is sequence of signs $\varepsilon_1, \ldots, \varepsilon_r$ such that for all $m, 1 \le m \le r$ and all sequences of indices $0 \le n_1 < n_2 < \ldots < n_m$ the Wronskians satisfy $\varepsilon_m W(\varphi_{n_1}, \varphi_{n_2}, \ldots, \varphi_{n_m})(x) > 0$ for all $x \in I$.

Examples of sign regular kernels

- $K(x,y)=x^y$ is well-known (and easily seen via Vandermonde determinant) to be STP_∞ on $(0,\infty)\times(-\infty,\infty)$. This recovers the case of ratio of two power series and the ratio of Mellin transforms
$$F(y)=\frac{\int_0^\infty x^{y-1}A(x)dx}{\int_0^\infty x^{y-1}B(x)dx}.$$
If $A(x)/B(x)$ is unimodal, then so is $F(y)$ and the monotonicity pattern of $A(x)/B(x)$ is preserved by $F(y)$ when it is not monotonic.

Examples of sign regular kernels

- The kernel $K(x,y)=\exp(xy)\in STP_\infty$ on $(-\infty,\infty)\times(-\infty,\infty)$. Hence, our theorem is applicable to the ratio of the Dirichlet series:

$$F(x)=\frac{\sum_{k=0}^{\infty}a_k e^{\lambda_k x}}{\sum_{k=0}^{\infty}b_k e^{\lambda_k x}},$$

where $\lambda_0<\lambda_1<\cdots$ If $F(x)$ is not monotonic it inherits monotonicity pattern of $\{a_k/b_k\}$. In a similar fashion, the ratios of two-sided and one-sided Laplace transforms (by restricting domain of integration to $(-\infty,0)$ and changing variable $x\to -x$)

$$F(y)=\frac{\int_{-\infty}^{\infty}e^{xy}A(x)dx}{\int_{-\infty}^{\infty}e^{xy}B(x)dx}\quad\text{and}\quad G(y)=\frac{\int_{0}^{\infty}e^{-xy}A(x)dx}{\int_{0}^{\infty}e^{-xy}B(x)dx}$$

satisfy our theorem, thus recovering some of the results of Yang and Tian (2019). Note that for the kernel $K(x,y)=e^{-xy}$ has signature $(+,-,-)$ so that monotonicity pattern of $A(x)/B(x)$ is preserved by $G(y)$ when it is not monotonic.

Examples of sign regular kernels

- The kernel $K(x,y)=(x+y)^{-\alpha}\in STP_\infty$ on $(0,\infty)\times(0,\infty)$ for each $\alpha>0$ (Carlson-Gustafson, 1983), so that our theorem is applicable to the ratio of the series

$$F(x)=\frac{\sum_{k=0}^{\infty}a_k(x+k)^{-\alpha}}{\sum_{k=0}^{\infty}b_k(x+k)^{-\alpha}}$$

and the ratio of the generalized Stieltjes transforms

$$F(y)=\frac{\int\limits_0^\infty \frac{A(x)}{(x+y)^\alpha}dx}{\int\limits_0^\infty \frac{B(x)}{(x+y)^\alpha}dx}.$$

If $A(x)/B(x)$ is unimodal, then so is $F(y)$ and the monotonicity pattern of $A(x)/B(x)$ is preserved by $F(y)$ when it is not monotonic.

Examples of sign regular kernels

- The kernel $K(x,y)=\Gamma(x+y)\in STP_\infty$ on $(0,\infty)\times(0,\infty)$. Hence, the kernel $K(n,x)=(x)_n=\Gamma(x+n)/\Gamma(x)\in STP_\infty$ on $\mathbb{N}_0\times(0,\infty)$ and monotonicity pattern of $\{a_k/b_k\}$ is preserved by the ratio
$$F(x)=\frac{\sum_{k=0}^{\infty}a_k(x)_k}{\sum_{k=0}^{\infty}b_k(x)_k}=\frac{A(x)}{B(x)}$$
if it is not monotonic. Note that
$$b_0^2F'(0^+)=b_0\sum_{k=1}^{\infty}a_k(k-1)!-a_0\sum_{k=1}^{\infty}b_k(k-1)!$$
As $x\to+\infty$: **Lemma.** Suppose $b_k>0$ for all $k\geq 0$ and
$$\frac{a_0}{b_0}\leq\frac{a_1}{b_1}\leq\cdots\leq\frac{a_m}{b_m},\quad \frac{a_m}{b_m}\geq\frac{a_{m+1}}{b_{m+1}}\geq\frac{a_{m+2}}{b_{m+2}}\cdots$$
and at least one inequality in each chain is strict. Assume that both series $A(x)$ and $B(x)$ converge uniformly on all compact subsets of $\mathbb{R}$. Then $F(x+1)-F(x)<0$ for all sufficiently large x.

- The kernel $K(x,y) = 1/\Gamma(x+y) \in SSR_\infty$ on $(0,\infty)\times(0,\infty)$, so that the kernel $K(n,x) = 1/(x)_n \in SSR_\infty$ on $\mathbb{N}_0 \times (0,\infty)$. In particular, it is SR_3 with the signature $(+,-,-)$. Hence, our theorem is applicable to the ratio of inverse factorial series:

$$F(x) = \frac{\sum_{k=0}^{\infty} a_k/(x)_k}{\sum_{k=0}^{\infty} b_k/(x)_k}$$

and monotonicity pattern of $\{a_k/b_k\}$ is reversed by $F(x)$ if it is not monotonic. Note further that

$$b_0^2 F'(x) = \frac{1}{x^2}(a_0 b_1 - a_1 b_0) + O(1/x^3) \text{ as } x \to +\infty \text{ and}$$

$$\left(\sum_{k=1}^{\infty} \frac{b_k}{(k-1)!}\right)^2 F'(0+) =$$

$$\sum_{k=1}^{\infty} \frac{b_0 b_k}{(k-1)!}\left(\frac{a_0}{b_0} - \frac{a_k}{b_k}\right) + \sum_{k=1}^{\infty}\sum_{j=1}^{k-1} \frac{b_k b_j (H_{j-1} - H_{k-1})}{(k-1)!(j-1)!}\left(\frac{a_k}{b_k} - \frac{a_j}{b_j}\right).$$

Examples of sign regular kernels

- The (Hankel) kernel $K(x,y)=\Gamma(c+x+y)/\Gamma(d+x+y)\in STP_\infty$ on $(0,\infty)\times(0,\infty)$ if $d>c>0$ so that the kernel $K(n,x)=(c+x)_n/(d+x)_n\in STP_\infty$ on $\mathbb{N}_0\times(0,\infty)$. Hence, our theorem is applicable to the ratio of the series:

$$F(x)=\frac{\sum_{k=0}^\infty a_k(c+x)_k/(d+x)_k}{\sum_{k=0}^\infty b_k(c+x)_k/(d+x)_k},$$

and monotonicity pattern of $\{a_k/b_k\}$ is inherited by $F(x)$.
Numerical experiments suggest that the kernel

$$K(x,y)=\frac{\Gamma(c+x+y)}{\Gamma(d+x+y)}$$

is also SR_∞ when $c>d$. In particular, it is SR_3 with the signature $(+,-,-)$. More generally we propose:
Conjecture. Suppose $K_i(x,y)=F_i(x+y)$, $i=1,2$, are SR_∞ on $(0,\infty)\times(0,\infty)$. Then the kernel $K(x,y)=K_1(x,y)K_2(x,y)$ is also SR_∞ on $(0,\infty)\times(0,\infty)$.

Examples of sign regular kernels

- Suppose $\mathbf{a} > 0$ and $\mathbf{m} = (m_1, m_2, \ldots, m_p)$ comprises non-negative integers. Then the kernel

$$K_1(x,y) = {}_pF_p\left(\begin{matrix}\mathbf{a}\\ \mathbf{a}+\mathbf{m}\end{matrix}\middle| xy\right) \in STP_\infty \text{ on } \mathbb{R}^2.$$

(Richards,1990). For any positive $\mathbf{a}$, $\mathbf{b}$ the kernel

$$K_2(x,y) = {}_pF_q\left(\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\middle| xy\right) \in STP_\infty \text{ on } (0,\infty)\times(0,\infty).$$

This implies that the main theorem is applicable to the ratios of the integral transforms of the form

$$F(x) = \frac{\int_{-\infty}^{\infty} K_1(x,t)A(t)w(t)dt}{\int_{-\infty}^{\infty} K_1(x,t)B(t)w(t)dt}, \quad G(x) = \frac{\int_0^{\infty} K_2(x,t)A(t)w(t)dt}{\int_0^{\infty} K_2(x,t)B(t)w(t)dt}$$

and the monotonicity pattern of $A(x)/B(x)$ is preserved both by $F(x)$ and $G(x)$ when they are not monotonic.

Examples of sign regular kernels

- Conjecture: $K(x,y)=\Gamma_q(x+y)\in STP_\infty$, $0<q<1$, where

$$\Gamma_q(z)=(1-q)^{1-z}\frac{(q;q)_\infty}{(q^z;q)_\infty},\quad (a;q)_n=\prod_{j=0}^{n-1}(1-aq^j).$$

  Proved: $K(n,x)=(q^x;q)_n\in STP_3$ on $\mathbb{N}_0\times(0,\infty)$, so that our theorem is applicable to the ratio of q-factorial series:

$$F(x)=\frac{\sum_{k=0}^\infty a_k(q^x;q)_k}{\sum_{k=0}^\infty b_k(q^x;q)_k}.$$

- Conjecture: $K(x,y)=1/\Gamma_q(x+y)\in SSR_\infty$, $0<q<1$. In progress: $K(n,x)=1/(q^x;q)_n\in SSR_3$ on $\mathbb{N}_0\times(0,\infty)$ with the sign pattern $(+,-,-)$, so that our theorem is applicable to the ratio of the inverse q-factorial series:

$$F(x)=\frac{\sum_{k=0}^\infty a_k/(q^x;q)_k}{\sum_{k=0}^\infty b_k/(q^x;q)_k}.$$

Examples of sign regular kernels

- Consider the kernel $K(n,x)=I_n(x)$, where I_ν is the modified Bessel function. According to (V.M. Buchstaber, A.A. Glutsyuk, 2019) this kernel is STP_∞ on $\mathbb{N}_0\times(0,\infty)$. Hence, the main theorem is applicable to the ratio of the modified Bessel function expansions of the form

$$F(x)=\frac{\sum_{k=0}^{\infty}a_kI_k(x)}{\sum_{k=0}^{\infty}b_kI_k(x)}$$

Similarly, restricting the interval I to non-negative integers, we conclude that

$$G(k)=\frac{\int_0^{\infty}I_k(t)A(t)w(t)dt}{\int_0^{\infty}I_k(x)B(t)w(t)dt}$$

is unimodal on $\mathbb{N}_0$ if so is $A(x)/B(x)$. If $F(x)$ ($G(k)$) is not monotonic it inherits the monotonicity pattern of $\{a_k/b_k\}$ ($A(x)/B(x)$).

- Let $L_j^\alpha(t)$ be the generalized Laguerre polynomials with $\alpha > -1$ and $j = 0, 1, \ldots$. According to (J. Delgado, H. Orera, J.M. Peña, 2019) the matrix $[L_j^\alpha(t_i)]_{0\le i,j\le n}$ is STP for each $n \in \mathbb{N}$ and $\alpha > -1$ if the points t_i satisfy $t_n < t_{n-1} < \cdots < t_1 < t_0 < 0$. This implies that ordering the points ascending $\hat{t}_i = t_{n-i}$ the kernel $K(j,t) = L_j^\alpha(t)$ is SSR_∞ with the sign pattern $(-1)^{r(r-1)/2}$ (where r is the size of the corresponding minor), in particular, it is SSR_3 with the sign pattern $(+,-,-)$. Hence, the main theorem is applicable to the ratio of the expansions in the Laguerre polynomials when the argument is outside of the interval of orthogonality:
$$F(x) = \frac{\sum_{k=0}^\infty a_k L_k^\alpha(x)}{\sum_{k=0}^\infty b_k L_k^\alpha(x)}$$
where $\alpha > -1$, $x < 0$, and if $F(x)$ is not monotonic it inherits the monotonicity pattern of $\{a_k/b_k\}$.

An application: hypergeometric ratios

$$F(\mu) := \frac{{}_pF_q\left(\begin{matrix}\mu, \mathbf{a}\\ \mathbf{b}\end{matrix}\middle| x\right)}{{}_sF_t\left(\begin{matrix}\mu, \mathbf{c}\\ \mathbf{d}\end{matrix}\middle| x\right)} = \frac{\sum_{k=0}^{\infty} f_k x^k (\mu)_k}{\sum_{k=0}^{\infty} g_k x^k (\mu)_k},$$

where (we use shorthand notation $(\mathbf{a})_k = (a_1)_k \cdots (a_{p-1})_k$)

$$\frac{f_k}{g_k} = \frac{(\mathbf{a})_k/(\mathbf{b})_k}{(\mathbf{c})_k/(\mathbf{d})_k} = \frac{(\mathbf{a})_k(\mathbf{d})_k}{(\mathbf{b})_k(\mathbf{c})_k}.$$

Elementary fact: log-concavity/log-convexity implies unimodality.

Log-concavity reduces to the inequality (similarly for log-convexity)

$$\frac{(\mathbf{a}+k-1)_1(\mathbf{d}+k-1)_1}{(\mathbf{b}+k-1)_1(\mathbf{c}+k-1)_1} \geq \frac{(\mathbf{a}+k)_1(\mathbf{d}+k)_1}{(\mathbf{b}+k)_1(\mathbf{c}+k)_1} \text{ for } k = 1, 2, \ldots$$

An application: hypergeometric ratios

Sufficient condition for this is the decrease of the rational function

$$R_{m,n}(x) = \frac{(\mathbf{a}+x)_1(\mathbf{d}+x)_1}{(\mathbf{b}+x)_1(\mathbf{c}+x)_1} = \frac{\prod_{k=1}^{m}(h_k+x)}{\prod_{k=1}^{n}(f_k+x)}$$

be monotone decreasing on $(0,\infty)$. Here $m=p+t$, $n=q+s$, $\mathbf{h}=(\mathbf{a},\mathbf{d})$, $\mathbf{f}=(\mathbf{b},\mathbf{c})$. Let $e_j(\mathbf{h})=e_j(h_1,\ldots,h_m)$ denote the j-th elementary symmetric polynomial. We have a slight extension of Biernaki–Krzyż as follows

Lemma (Kalmykov-K., 2017)

If $m \le n$ and

$$\frac{e_n(\mathbf{f})}{e_m(\mathbf{h})} \le \frac{e_{n-1}(\mathbf{f})}{e_{m-1}(\mathbf{h})} \le \cdots \le \frac{e_{n-m+1}(\mathbf{f})}{e_1(\mathbf{h})} \le e_{n-m}(\mathbf{f}),$$

then the function $R_{m,n}(x)$ is monotone decreasing on $(0,\infty)$. These inequalities hold, in particular, if $0<h_1\le h_2\le\cdots\le h_m$, $0<f_1\le f_2\le\cdots\le f_m$ and $\sum_{j=1}^{k} f_j \le \sum_{j=1}^{k} h_j$ for $k=1,\ldots,m$.

An application: hypergeometric ratios

Theorem (K.-Vishnyakova-Zhang, 2024)

Suppose $p + t \leq q + s$ and the parameter vectors $\mathbf{h} := (\mathbf{a}, \mathbf{d})$, $\mathbf{f} := (\mathbf{b}, \mathbf{c})$ satisfy the above Lemma. Then

$$F(\mu) = {}_pF_q\left(\begin{matrix}\mu, \mathbf{a}\\ \mathbf{b}\end{matrix}\middle| x\right) \Big/ {}_sF_t\left(\begin{matrix}\mu, \mathbf{c}\\ \mathbf{d}\end{matrix}\middle| x\right)$$

is unimodal for each positive x in the domain of convergence. In the case when it is not monotone, it is first increasing and then decreasing.

Note that the sign of $F'(0)$ coincides with that of

$$\frac{(\mathbf{a})_1}{(\mathbf{b})_1}\,{}_{p+1}F_{q+1}\left(\begin{matrix}1, \mathbf{a}+1\\ 2, \mathbf{b}+1\end{matrix}\middle| x\right) - \frac{(\mathbf{c})_1}{(\mathbf{d})_1}\,{}_{s+1}F_{t+1}\left(\begin{matrix}1, \mathbf{c}+1\\ 2, \mathbf{d}+1\end{matrix}\middle| x\right).$$

Hence, if this quantity is negative, then $F(\mu)$ is monotone decreasing; if it is positive, $F(\mu)$ is it first increasing and then decreasing.

More hypergeometric and q-hypergeometric ratios

In a similar fashion we get conditions for unimodality of the hypergeometric ratios

$$\mu \to {}_pF_q\left(\begin{matrix}\mathbf{a}\\ \mu,\mathbf{b}\end{matrix}\middle|x\right)\Big/{}_sF_t\left(\begin{matrix}\mathbf{c}\\ \mu,\mathbf{d}\end{matrix}\middle|x\right),$$

$$\mu \to {}_pF_q\left(\begin{matrix}\mathbf{c}+\mu,\mathbf{a}_1\\ \mathbf{d}+\mu,\mathbf{b}_1\end{matrix}\middle|x\right)\Big/{}_sF_t\left(\begin{matrix}\mathbf{c}+\mu,\mathbf{b}_2\\ \mathbf{d}+\mu,\mathbf{a}_2\end{matrix}\middle|x\right),$$

and q-hypergeometric ratios

$$\mu \to {}_p\varphi_q\left(\begin{matrix}q^{\mu},q^{\mathbf{a}_1}\\ q^{\mathbf{b}_1}\end{matrix}\middle|q;x\right)\Big/{}_s\varphi_t\left(\begin{matrix}q^{\mu},q^{\mathbf{b}_2}\\ q^{\mathbf{a}_2}\end{matrix}\middle|q;x\right),$$

$$\mu \to {}_p\varphi_q\left(\begin{matrix}q^{\mathbf{a}_1}\\ q^{\mu},q^{\mathbf{b}_1}\end{matrix}\middle|q;x\right)\Big/{}_s\varphi_t\left(\begin{matrix}q^{\mathbf{b}_2}\\ q^{\mu},q^{\mathbf{a}_2}\end{matrix}\middle|q;x\right).$$

An application: Nuttall's Q function

Nuttall Q-function (important in communication theory) is defined by

$$Q_{\mu,\nu}(a,b) = \int_b^\infty x^\mu e^{-(x^2+a^2)/2} I_\nu(ax)dx$$

with $a > 0$, $b \geq 0$, $\nu > -1$, $\mu > 0$ and I_ν standing for the modified Bessel function of the first kind. We have

Theorem (K.-Vishnyakova-Zhang, 2024)

Suppose $b \geq 0$, $\nu_1 - \nu_2 = 2\ell \in \mathbb{N}$ and $0 < a_1 \leq a_2$. Then the ratio

$$\mu \to \frac{Q_{\mu,\nu_1}(a_1,b)}{Q_{\mu,\nu_2}(a_2,b)} \text{ is unimodal on } (0,\infty).$$

Conjecture. Suppose $b \geq 0$, $\nu_1 \geq \nu_2 > -1$ and $0 < a_1 \leq a_2$. Then the ratio

$$x \to \frac{I_{\nu_1}(a_1 x)}{I_{\nu_2}(a_2 x)} \text{ is unimodal on } (0,\infty).$$

Moreover, if $\nu_1 \geq \nu_2 > 0$ it is log-concave.

THANK YOU FOR ATTENTION!