Lipschitz Permutations on Certain Graphs

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Abstract

Let $G(V, E)$ be a graph with n vertices. An integer valued function f on V is called K-Lipschitz if it changes by at most K along edges. Random K-Lipschitz functions are studied with respect to their typical behaviour. Here we impose a certain constraint and consider only K-Lipschitz permutations. We say that G admits K-Lipschitz permutation if there exists a bijective mapping $f : V(G) \to [1..n]$ that is K-Lipschitz. We estimate the minimum K for which G admits a K-Lipschitz permutation. Let us denote

$$K(G) := \inf_{K\in\mathbb{R}}\{K \in \mathbb{R} : G \text{ admits } K\text{-Lipschitz permutation. }\}$$

Our goal is to estimate $K(G)$ on certain class of graphs.

Claim 1. Let T be a tree with n vertices and maximum degree at most d. Then

$$C_1\frac{n}{\log n} \le K(T) \le C_2\frac{n}{\log n} \qquad (1),$$

where $C_1, C_2 > 0$ are constants depending only on d, but not on n or T. We also prove a negative result that shows (1) cannot be extended to the class of all graphs with maximum degree d.

Claim 2. Let $d \in \mathbb{N}, d \ge 6$. Then for any $C > 0$ there exists $N \in \mathbb{N}$, such that for all $n \ge N$, there exists a graph G with n vertices and maximum degree at most d, such that

$$K(G) > C\frac{n}{\log n}.$$

The proof of Claim 2 is non-constructive. We use the probabilistic method to obtain the existence. So, the class of graphs $\mathcal{D}(d, n)$ of n vertices and maximum degree at most d is significantly wider with respect to the introduced property. It's interesting to find some more specific bounds on $K(\mathcal{D}(d, n))$.