Stability of dnoidal waves of the Schrodinger-KdV system

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- Schrodinger-Korteweg-de Vries system

$$\begin{cases} iu_t + u_{xx} + uv = 0 \\ v_t + v_x + \beta(v^2)_x + \alpha v_{xxx} = -\frac{1}{2}(|u|^2)_x. \end{cases} \tag{1}$$

- periodic waves
  We take the ansatz $u(t,x) = e^{i\omega t} e^{i\frac{c}{2}(x-ct)}\varphi(x-ct)$, $v(t,x) = \psi(x-ct)$,

$$\begin{aligned} &\varphi'' - \sigma\varphi + \varphi\psi = 0 \\ &-c\psi' + \psi' + 2\beta\psi\psi' + \alpha\psi''' = -\tfrac{1}{2}(\varphi^2)', \end{aligned} \tag{2}$$

  where $\sigma = \omega - \frac{c^2}{4}$.

Theorem

Let $c > 1, \omega \in \mathbf{R} : \omega > \frac{c^2}{4}$*, and so* $\sigma := \omega - \frac{c^2}{4} > 0$*. If* $\alpha : 0 < \alpha < \frac{c-1}{4\sigma}$ *and* $\beta = 3\alpha$*, then the system* (2) *has a one parameter family of solutions in the form* $(e^{i\omega t} e^{i\frac{c}{2}(x-ct)}\varphi(x-ct), \psi(x-ct))$*, with*

$$\varphi(x) = \varphi_0 dn(\gamma x, \kappa), \psi(x) = A\varphi^2(x, \kappa), \kappa \in (0,1) \tag{3}$$

where

$$A = \frac{1}{2(c-1-4\alpha\sigma)} > 0, \quad \gamma^2 = \frac{\sigma}{2-\kappa^2}, \quad \varphi_0^2 = \frac{2\gamma^2}{A},$$

$$u(t,x) = e^{i\omega t} e^{i\frac{c}{2}(x-ct)}(\varphi(x-ct)+p(t,x-ct)); \;\; v(t,x) = \psi(x-ct)+q(t,x$$

Plugging in the system and ignoring all quadratic and higher terms in p, q, we get

$$\begin{cases} ip_t - \sigma p + p_{xx} + \varphi q + \psi p = \\ q_t - cq_x + q_x + 2\beta(\psi q)_x + \alpha q_{xxx} + (\varphi \Re p)_x = 0. \end{cases} \quad (4)$$

Separating the real and imaginary part $p = p_1 + ip_2$, we get

$$\begin{cases} p_{1t} = -p_{2xx} - \sigma p_2 - \psi p_2 \\ -p_{2t} = -p_{1xx} + \sigma p_1 - \psi p_1 - \varphi q \\ q_t = \partial_x(-\alpha q_{xx} + cq - q - 2\beta\psi q - \varphi p_1). \end{cases} \quad (5)$$

For $\vec{U} = (q, p_1, p_2)$, the system (5) can be written in the form

$$\vec{U}_t = \mathcal{J}\mathcal{L}\vec{U}, \tag{6}$$

where

$$\mathcal{J} = \begin{pmatrix} \partial_x & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \mathcal{L} = \begin{pmatrix} \mathcal{L}_1 & -\varphi & 0 \\ -\varphi & \mathcal{L}_2 & 0 \\ 0 & 0 & \mathcal{L}_2 \end{pmatrix} \tag{7}$$

$$\mathcal{L}_1 = -\alpha\partial_x^2 + (c-1) - 2\beta\psi \tag{8}$$

$$\mathcal{L}_2 = -\partial_x^2 + \sigma - \psi. \tag{9}$$

Definition

We say that the wave φ is spectrally stable, if the associated eigenvalue problem

$$\mathcal{J}\mathcal{L}\vec{f} = \lambda\vec{f}, \tag{10}$$

does not have a non-trivial solution $(\lambda, \vec{f}) : \Re\lambda > 0, \vec{U} \neq 0, \vec{f} \in D(\mathcal{L})$. Otherwise, we say that the wave φ is spectrally unstable.

$P_{\{\xi\}^\perp}$, which projects onto the orthogonal complement of a fixed vector ξ. Specifically,

$$P_{\{\xi\}^\perp} f = f - \frac{\langle f, \xi \rangle}{\|\xi\|^2}\xi,$$

Lemma (Weinstein)

Let $(A, D(A))$ be a self-adjoint operator acting on a Hilbert space, with exactly one negative simple eigenvalue, say $-\sigma^2 : A\eta_0 = -\sigma^2\eta_0$. Assume also the zero gap condition

$$A|_{span\{\eta_0, Ker(A)\}^\perp} \geq \delta_0 > 0.$$

Assume that there is $\xi_0 \in Ker(A)^\perp$, so that $\langle A^{-1}\xi_0, \xi_0 \rangle < 0$. Then,

$$A|_{\{\xi_0\}^\perp} \geq 0. \tag{11}$$

Concretely, $P_{\{\xi_0\}^\perp} A P_{\{\xi_0\}^\perp} \geq 0$,

Theorem

The operator $\mathcal{L}_2, D(\mathcal{L}_2) = H^2_{per.}[-T, T]$ *is a non-negative operator. Also,*

$$Ker(\mathcal{L}_2) = span[\varphi], \;\; \mathcal{L}_2|_{\{\varphi\}^\perp} > 0.$$

Regarding the operator $L := \mathcal{L}_2 - 2A\varphi^2 = -\partial_x^2 + \sigma - 3A\varphi^2$, *we have the following spectral properties*

- L has exactly one negative eigenvalue, with a positive eigenfunction, say h_0.

- $L[\varphi'] = 0$. *Moreover* $Ker(L) = span[\varphi']$ *and* $L|_{\{h_0,\varphi'\}^\perp} > 0$.

- $L|_{\{\varphi\}^\perp} \geq 0$ *and in fact,* $L|_{\{\varphi,\varphi'\}^\perp} > 0$. *That is, there exists* $\delta > 0$, *so that*

$$\langle Lh, h\rangle \geq \delta \|h\|^2, \forall h : h \perp \varphi, h \perp \varphi'$$

Theorem

For all values of the parameters as stated in theorem 1, the scalar Schrödinger operator

$$\mathcal{Q} := \mathcal{L}_1 - \frac{1}{2A} = -\alpha\partial_x^2 + \left(c - 1 - \frac{1}{2A}\right) - 2\beta\psi, \quad D(\mathcal{Q}) = H^2_{per.}[-T, T],$$

has the following spectral properties

- $\mathcal{Q}$ has exactly one negative eigenvalue, with an eigenfunction, say $\psi_0 > 0$.

- $\mathcal{Q}[\psi'] = 0$. Moreover Ker$(\mathcal{Q})$ = span$[\psi']$ and $\mathcal{Q}|_{\{\psi_0,\psi'\}^\perp} > 0$.

- $\mathcal{Q}|_{\{\psi\}^\perp} \geq 0$ and moreover, $\mathcal{Q}|_{\{\psi,\psi'\}^\perp} > 0$. That is, there exists $\delta > 0$, so that

$$\langle \mathcal{Q}h, h\rangle \geq \delta\|h\|^2, : h \perp \psi, h \perp \psi'$$

We now work with the eigenvalue problem (10) directly. The aim is to extract some extra orthogonality relations, which will help in our goal to establish spectral stability. Specifically, we write (10)

$$\begin{cases} \partial_x(\mathcal{L}_1 f_1 - \varphi f_2) = \lambda f_1 \\ -(-\varphi f_1 + \mathcal{L}_2 f_2) = \lambda f_3 \\ \mathcal{L}_2 f_3 = \lambda f_2. \end{cases} \tag{12}$$

Specifically, we are looking to show that there is no non-trivial solution to (12), for any $\lambda > 0$. We assume, for a contradiction, there is such a solution $\vec{f}$ for some fixed $\lambda > 0$.

From these relations, by taking dot product with 1 in the first equation and φ in the third one, we immediately see that $f_1 \perp 1$ and $f_2 \perp \varphi$ (recall $\mathcal{L}_2\varphi = 0$). Clearly, since ∂_x annihilates constants, $\partial_x = \partial_x P_{\{1\}^\perp}$. Also, upon introducing $P_{\{\varphi\}^\perp}$ in the second and third equations (note that $P_{\{\varphi\}^\perp}\mathcal{L}_2 = \mathcal{L}_2$), we arrive at

$$\begin{cases} \partial_x P_{\{1\}^\perp}(\mathcal{L}_1 f_1 - \varphi f_2) = \lambda f_1 \\ -P_{\{\varphi\}^\perp}(-\varphi f_1 + \mathcal{L}_2 f_2) = \lambda \tilde{f}_3 \\ P_{\{\varphi\}^\perp}\mathcal{L}_2 \tilde{f}_3 = \lambda f_2. \end{cases} \tag{13}$$

where $\tilde{f}_3 = P_{\{\varphi\}^\perp} f_3 \perp \varphi$ (note that $\mathcal{L}_2 f_3 = \mathcal{L}_2 \tilde{f}_3$, since $f_3 - \tilde{f}_3 \in span[\varphi] = Ker(\mathcal{L}_2)$).

The form (13), yields an important restriction of the eigenvalue problem, which are useful in the sequel. Indeed, we may now rewrite (13)

$$\mathcal{J}\begin{pmatrix} P_{\{1\}^\perp} & 0 & 0 \\ 0 & P_{\{\varphi\}^\perp} & 0 \\ 0 & 0 & P_{\{\varphi\}^\perp} \end{pmatrix} \mathcal{L} \begin{pmatrix} P_{\{1\}^\perp} & 0 & 0 \\ 0 & P_{\{\varphi\}^\perp} & 0 \\ 0 & 0 & P_{\{\varphi\}^\perp} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ \tilde{f}_3 \end{pmatrix} = \lambda \begin{pmatrix} f_1 \\ f_2 \\ \tilde{f}_3 \end{pmatrix}$$

So, it makes sense to introduce the restricted self-adjoint operator

$$\mathcal{H} := \begin{pmatrix} P_{\{1\}^\perp} & 0 & 0 \\ 0 & P_{\{\varphi\}^\perp} & 0 \\ 0 & 0 & P_{\{\varphi\}^\perp} \end{pmatrix} \mathcal{L} \begin{pmatrix} P_{\{1\}^\perp} & 0 & 0 \\ 0 & P_{\{\varphi\}^\perp} & 0 \\ 0 & 0 & P_{\{\varphi\}^\perp} \end{pmatrix}.$$

The goal now is to disprove that the eigenvalue problem

$$\mathcal{J}\mathcal{H}\vec{f} = \lambda\vec{f} \qquad (14)$$

with $\lambda > 0$, does not have any non-trivial solutions.

Theorem

The operator $\mathcal{H}$ is non-negative.

It is now trivial to finish the spectral stability proof for the dnoidal waves.

Theorem

The eigenvalue problem (14) *does not have non-trivial solutions with $\Re\lambda > 0$.*

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Theorem

The eigenvalue problem (14) *does not have non-trivial solutions with $\Re\lambda > 0$.*

Thank you for Attention