Resonance Cases for Nonlocal Wave Equation

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Boundary Value Problem

$$u_{tt} = u_{xx}, \quad 0 < x < 1, \quad 0 < t, \tag{1}$$

$$u(x,0) = f(x), \quad u_t(x,0) = 0, \quad 0 \leq x \leq 1, \tag{2}$$

$$u(0,t) = 0, \quad u(1,t) = 0 \quad 0 \leq t. \tag{3}$$

Its solution in the series form is given by

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin n\pi x \cos n\pi t$$

where A_n are the Fourier coefficients of the expansion of $f(x)$ in terms of the sine functions $\sin n\pi x$, $n = 1, 2, .$ The solution is a periodic function of time with period 1 and, hence, is bounded.

Nonlocal Boundary Value Problems

$$\begin{aligned} u_{tt} &= u_{xx}, \quad 0 < x < 1, \quad 0 < t, \\ u(x,0) &= f(x), \quad u_t(x,0) = 0, \\ u(0,t) &= 0, \quad \int_0^1 u(x,t)dx = 0. \end{aligned} \tag{4}$$

Beilin, S.A. Existence of solutions for one-dimensional wave equations with nonlocal conditions. *Electron. J. Diff.Eqns.*, vol.2001 no. **76** (2001), 1 - 8.

$$\begin{aligned} u_{tt} &= u_{xx}, \quad 0 < x < 1, \quad 0 < t \\ u(x,0) &= f(x), \quad u_t(x,0) = 0, \\ u(0,t) &= 0, \quad u(1,t) = u(c,t). \end{aligned}$$

Our aim is to find explicit solutions.

$$\Phi_\xi\{y(\xi)\} = \int_0^1 y(\xi)d\xi = 0$$

Ionkin, N.I., Solution of One Boundary-Value Problem of Heat Conduction Theory with a Nonclassical Boundary Condition, Differ. Uravn., 1977, vol. 13, no. 2, pp. 294-304.

$$\Phi_\xi\{y(\xi)\} = y(1) - y(c), \quad c \in (0, 1).$$

A.V. Bitsadze, A.A. Samarskii, On some simplest generalizations of linear elliptic boundary value problems, *Doklady AN SSSR*, **185**, No 4 (1969), 739-741 (In Russian).

With BVP (4) it is connected the following non-local eigenvalue problem in $C^2([0,1])$:

$$y'' + \lambda^2 y = f(x), \quad y(0) = 0, \quad \Phi\{y\} = \int_0^1 y(\xi)d\xi = 0, \quad x \in [0,1]. \tag{5}$$

The sine indicatrix of the functional Φ is

$$E(\lambda) = \Phi\{\frac{\sin \lambda\xi}{\lambda}\} = \int_0^1 \frac{\sin \lambda\xi}{\lambda} d\xi = \frac{1 - \cos \lambda}{\lambda^2}.$$

The eigenvalues $\lambda : E(\lambda) = 0$ are

$$\lambda_n = 2n\pi, \quad n = 1, 2, 3, \ldots.$$

and each of them has multiplicity 2.

The functions

$$\sin \lambda_n \xi, \qquad x \cos \lambda_n \xi$$

are eigenfunctions and associated eigenfunctions, respectively.

The spectral projections has the representation:

$$P_{\lambda_n}\{f\} = 4\left(\int_0^a f(\xi)(a-\xi)\sin \lambda_n \xi d\xi\right)\sin \lambda_n x - 4\left(\int_0^a f(\xi)(1-\cos \lambda_n \xi)d\xi\right)x\cos \lambda_n x$$

Solution of the nonlocal boundaryvalue problem (4) in a series form was found by

Beilin, S.A. Existence of solutions for one-dimensional wave equations with nonlocal conditions. *Electron. J. Diff.Eqns.*, vol.2001 no. **76** (2001), 1 - 8.

$$u(x,t)=\sum_{n=1}^{\infty}(A_n(t)\sin\lambda_n x+B_n(t)x\cos\lambda_n x)$$

$$
\begin{aligned}
&u_{tt} = u_{xx}, \quad 0 < x < 1, \quad 0 < t \\
&u(x,0) = f(x), \quad u_t(x,0) = 0, \\
&u(0,t) = 0, \quad u(1,t) = u(c,t).
\end{aligned} \tag{6}
$$

With BVP (6) it is connected the following non-local eigenvalue problem in $C^2([0,1])$:

$$y'' + \lambda^2 y = f(x), \quad y(0) = 0, \quad \Phi\{y\} = y(1) - y(c) = 0, \quad x \in [0,1]. \tag{7}$$

The sine indicatrix of the functional Φ is

$$E(\lambda) = \Phi\{\frac{\sin \lambda\xi}{\lambda}\} = \frac{\sin \lambda - \sin c\lambda}{\lambda}.$$

The eigenvalues $\lambda : E(\lambda) = 0$ are

$$\lambda_n = \frac{(2n-1)\pi}{1+c}, \quad \mu_k = \frac{2k\pi}{1-c}, \qquad n, k \in \mathbb{N}$$

1) The arithmetic progressions (λ_n) and (μ_k) have no common terms. This happens when c is an irrational number.

2) For some rational c it may happen some λ_n to be equal to some μ_k. For example, $c = \frac{1}{5}$, $c = \frac{3}{7}$.

1. Let all the eigenvalues be simple. Then the spectral Riesz' projectors for (λ_n) are

$$P_{\lambda_n}\{f\} =$$

$$= \frac{4}{\cos\lambda_n - c\cos c\lambda_n}\left(\int_0^1 \sin(\lambda_n(1-\xi))f(\xi)d\xi - \int_0^c \sin(\lambda_n(c-\xi))f(\xi)d\xi\right)\sin(\lambda_n x)$$

and the spectral projectors for (μ_k) are

$$P_{\mu_k}\{f\} =$$

$$= \frac{4}{\cos\mu_k - c\cos c\mu_k}\left(\int_0^1 \sin(\mu_k(1-\xi))f(\xi)d\xi - \int_0^c \sin(\mu_k(c-\xi))f(\xi)d\xi\right)\sin(\mu_k x).$$

2. If $\lambda_n = \mu_k$, then $E(\lambda_n) = 0$, $E'(\lambda_n) = 0$ but $E''(\lambda_n) \neq 0$. In this case the spectral projectors are

$$P_{\lambda_n}\{f\} = C_n \left(\int_0^1 f(\xi) \sin \lambda_n (1-\xi) d\xi - \int_0^c f(\xi) \sin \lambda_n (c-\xi) d\xi \right) \quad x \cos \lambda_n x$$
$$+ \left[C_n \left(\int_0^1 (1-\xi) f(\xi) \cos \lambda_n (1-\xi) d\xi - \int_0^c (c-\xi) f(\xi) \cos \lambda_n (c-\xi) d\xi \right) \right.$$
$$\left. + \frac{G_n - C_n}{\lambda_n} \left(\int_0^1 f(\xi) \sin \lambda_n (1-\xi) d\xi - \int_0^c f(\xi) \sin \lambda_n (c-\xi) d\xi \right) \right] \quad \sin \lambda_n x,$$

where

$$C_n = \frac{4}{(1-c^2)\sin \lambda_n}, \quad G_n = \frac{4(\lambda_n \cos \lambda_n - c^3 \lambda_n \cos \lambda_n c - 3(1-c^2)\sin \lambda_n)}{3(1-c^2)^2 \sin^2 \lambda_n}.$$

Definition 1. Let $f \in C[0,1]$. The formal spectral expansion of $f(x)$ for the eigenvalue problem

$$\frac{d^2}{dx^2}y(x) + \lambda^2 y(x) = 0, \quad 0 < x < 1, \tag{8}$$
$$y(0) = 0, \qquad \Phi_\xi\{y(\xi)\} = 0,$$

is the correspondence

$$f(x) \sim \sum_{k=1}^{\infty} P_{\lambda_k}. \tag{9}$$

In fact, this formal spectral expansion, is not completely formal, since it has a uniqueness property: if $P_{\lambda_k}\{f\} = 0$ for $k = 1, 2, ...$, then $f(x) \equiv 0$.

This follows immediately from a theorem of N. Bozhinov.

Bozhinov N. S. On theorems of uniqueness and completeness of expansion on eigen and associated eigenfunctions of the nonlocal Sturm-Liouville operator on a finite interval. *Diferenzialnye Uravnenya*, **26**, 5, 1990, pp. 741-453 (Russian)

The operation

$$(f \overset{x}{*} g)(x) = -\frac{1}{2}\widetilde{\Phi}_\xi\{h(x,\xi)\}, \tag{10}$$

where

$$h(x,\zeta) = \int_x^\zeta f(\zeta + x - \eta)g(\eta)d\eta - \int_{-x}^\zeta f(|\zeta - x - \eta|)g(|\eta|)\mathrm{sgn}(\eta(\zeta - x - \eta))d\eta$$

and $\widetilde{\Phi}_\xi = \Phi_\xi \circ l_\xi$, $l_\xi f(\xi) = \int_0^\xi f(\zeta)d\zeta$
is a bilinear, commutative and associative operations in C([0; 1]).

I.H. Dimovski, *Convolutional Calculus*, Kluwer, Dordrecht 1990.

We can represent the solutions of the problem (6)

$$u_{tt} = u_{xx}, \quad 0 < x < 1, \quad 0 < t$$
$$u(x,0) = f(x), \quad u_t(x,0) = 0,$$
$$u(0,t) = 0, \quad u(1,t) = u(c,t).$$

$$u = \frac{\partial^4}{\partial x^4}(\Omega * f)$$

Where $\Omega(x,t)$ is the solution of (6) for

$$f(x) = \frac{x^3}{6} - \frac{1+c+c^2}{6}x.$$

$f(x)$ is the unique polynomial of degree 3 such that $f''(x) = x$ and $f(0) = f(1) - f(c) = 0$.

We can represent the solutions of the problem (4)

$$\begin{aligned} &u_{tt} = u_{xx}, \quad 0 < x < 1, \quad 0 < t, \\ &u(x,0) = f(x), \quad u_t(x,0) = 0, \\ &u(0,t) = 0, \quad \int_0^1 u(x,t)dx = 0. \end{aligned}$$

$$u = \frac{\partial^4}{\partial x^4}(U * f)$$

Where $U(x,t)$ is the solution of (4) for

$$f(x) = \frac{x^3}{6} - \frac{x}{12}.$$

$f(x)$ is the unique polynomial of degree 3 such that $f''(x) = x$ and

$$f(0) = \int_0^1 f(x)dx = 0$$

.

It is observed that in this case the solution has the form

$$u(x,t) = v(x,t) + t \quad w(x,t)$$

where v and w are continuous functions in $0 < x < 1, \quad 0 < t$, periodic with respect to t, and as such they are uniformly bounded. This means that the oscillation amplitude of the system described by this equation increases infinitely with time at any fixed $x \in [0,1]$ in absence of external forces ($F = 0$).

THANK YOU FOR YOUR ATTENTION