Bernstein functions and Jeffreys-type equations

Emilia Bazhlekova

Institute of Mathematics and Informatics
Bulgarian Academy of Sciences

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Complete Bernstein functions

$\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, $\varphi \in C^\infty$

$\varphi \in \mathcal{CMF}$ (**completely monotone functions**):

$$(-1)^n \varphi^{(n)}(t) \geq 0, \quad t > 0, n = 0, 1, 2, ... \tag{1}$$

$\varphi \in \mathcal{BF}$ (**Bernstein functions**) $\Leftrightarrow \varphi \geq 0$, $\varphi' \in \mathcal{CMF}$

$\varphi \in \mathcal{CBF}$ (**complete Bernstein functions**):

$$\varphi(s) = a + bs + \int_0^\infty (1 - e^{-s\tau}) m(\tau)\, d\tau, \tag{2}$$

where $a, b \geq 0$, $m \in \mathcal{CMF}$; $\mathcal{CBF} \subset \mathcal{BF}$.

Basic examples: $\mathcal{CBF}$: s^α, $s^\alpha/(s+a)^\alpha$, $\alpha \in [0,1]$, $a > 0$; $\log(1+s)$;
$\mathcal{CMF}$: $\exp(-t)$; $t^{\beta-1} E_{\alpha,\beta}(-\lambda t^\alpha) \in \mathcal{CMF}$ for $\lambda > 0$, $0 < \alpha \leq \beta \leq 1$;

$$E_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + \beta)} \ - \ \text{Mittag-Leffler function} \tag{3}$$

[1] R.L. Schilling, R. Song, Z. Vondraček (2010), Bernstein functions: Theory and applications.

Complete Bernstein functions

All classes are convex cones (closed under linear combinations with positive coefficients); $\mathcal{CMF}$ is closed under multiplication.

Define the set:

$$\mathcal{CBF}^{\delta} := \left\{\varphi^{\delta} \,:\, \varphi \in \mathcal{CBF}\right\}, \qquad \delta > 0. \tag{4}$$

Property: $\mathcal{CBF}^{\delta_1} \subseteq \mathcal{CBF}^{\delta}, \qquad 0 < \delta_1 \leq \delta.$

Proposition 1: Let $f_1 \in \mathcal{CBF}^{\delta_1}, f_2 \in \mathcal{CBF}^{\delta_2}$, where $\delta_1, \delta_2 > 0$. Then

$$f_1 \cdot f_2 \in \mathcal{CBF}^{\delta_1+\delta_2}. \tag{5}$$

Proposition 2: Let $\delta \in (0,1]$. Then the set $\mathcal{CBF}^{\delta}$ is a convex cone.

Proposition 3: Let $0 \leq \delta_n \leq \delta \leq 1$, $q_n \geq 0$, $n = 1, \ldots, N$. Then

$$s^{\delta} + \sum_{n=1}^{N} q_n s^{\delta_n} \in \mathcal{CBF}^{\delta} \quad \text{and} \quad \left(s^{-\delta} + \sum_{n=1}^{N} q_n s^{-\delta_n}\right)^{-1} \in \mathcal{CBF}^{\delta}.$$

Fractional Jeffreys-type diffusion-wave equation

$$\left(1 + aD_t^{\alpha}\right) \frac{\partial}{\partial t} u(\mathbf{x}, t) = D_t^{1-\gamma} \left(1 + bD_t^{\beta}\right) \Delta u(\mathbf{x}, t), \tag{6}$$

where $0 < \alpha, \beta, \gamma \leq 1$, $a > 0, b > 0$, Δ - Laplace operator, and D_t^{δ} - the Riemann-Liouville fractional derivative of order $\delta > 0$

$$D_t^{\delta} f(t) = \frac{d^m}{dt^m} \int_0^t \frac{(t-\tau)^{m-\delta-1}}{\Gamma(m-\delta)} f(\tau)\, d\tau, \quad m-1 < \delta < m,\ m = 1, 2, \ldots$$

[2] Awad, E., Sandev, T., Metzler, R., Chechkin, A.: From continuous-time random walks to the fractional Jeffreys equation: Solution and properties. *Int. J. Heat Mass Transf.* (2021)

Multi-term fractional Jeffreys-type diffusion-wave equation

Multi-term generalization:

$$\left(1 + aD_t^{\alpha} + \sum_{k=1}^{K} a_k D_t^{\alpha_k}\right)\frac{\partial u}{\partial t} = D_t^{1-\gamma}\left(1 + bD_t^{\beta} + \sum_{j=1}^{J} b_j D_t^{\beta_j}\right)\Delta u, \qquad (7)$$

with the following assumptions on the parameters

$$\begin{aligned} &0 < \alpha_k < \alpha \leq 1, \quad 0 < \beta_j < \beta \leq 1, \quad 0 < \gamma \leq 1, \\ &a > 0,\ b > 0, \quad a_k \geq 0,\ b_j \geq 0, \\ &k = 1, \dots, K,\ j = 1, \dots, J, \quad K, J \in \mathbb{N}. \end{aligned} \qquad (8)$$

Characteristic function

$$g(s) = s^{\gamma}\left(1 + as^{\alpha} + \sum_{k=1}^{K} a_k s^{\alpha_k}\right)\left(1 + bs^{\beta} + \sum_{j=1}^{J} b_j s^{\beta_j}\right)^{-1}, \quad s > 0. \qquad (9)$$

Aim: Find restrictions on the parameters, which guarantee $g(s) \in \mathcal{CBF}^2$

Restrictions on the parameters

Theorem 1: Assume $\beta \le \gamma$. Then $g(s) \in \mathcal{CBF}^{\alpha+\gamma}$.

Theorem 2: Assume $\alpha = \beta$, $K = J$, $\alpha_k = \beta_k$, $a_k, b_k > 0$, $k = 1, \ldots, K$, and $0 < \alpha_1 < \cdots < \alpha_K < \alpha < 1$. Then the characteristic function $g(s)$ obeys the inclusions:

(a) If $\gamma = 1$ and $1 \ge a_1/b_1 \ge \cdots \ge a_K/b_K \ge a/b$ then $g(s) \in \mathcal{CBF}$;

(b) If $1 \ge b_1/a_1 \ge \cdots \ge b_K/a_K \ge b/a$ then $g(s) \in \mathcal{CBF}^{1+\gamma}$.

Theorem 3: In the single-term case with $\alpha = \beta$ the characteristic function obeys the inclusions:

(a) If $0 < a < b$ and $\alpha \le \gamma$ then $g(s) \in \mathcal{CBF}^{\gamma}$;

(b) If $a > b > 0$ then $g(s) \in \mathcal{CBF}^{\alpha+\gamma}$.

Single-term case

Corollary: Let $0 < \alpha, \beta, \gamma \leq 1$ and $a > 0, b > 0$. The characteristic function of the single-term equation satisfies $g(s) \in \mathcal{CBF}$ if any of the following sets of conditions is satisfied:

(a) $\beta \leq \gamma$ and $\alpha + \gamma \leq 1$;

(b) $a < b$ and $\alpha = \beta \leq \gamma$;

(c) $a > b$, $\alpha = \beta$, and $\alpha + \gamma \leq 1$.

This corollary expands the conditions in [2].

Subordination results

X - Banach space with norm $\|.\|$; A - closed linear operator in X with dense domain $D(A) \subset X$, equipped with the graph norm.

Fractional Jeffreys-type equation in abstract form:

$$\left(1 + aD_t^{\alpha} + \sum_{k=1}^{K} a_k D_t^{\alpha_k}\right) u'(t) = D_t^{1-\gamma} \left(1 + bD_t^{\beta} + \sum_{j=1}^{J} b_j D_t^{\beta_j}\right) Au(t). \quad (10)$$

Initial conditions: $u(0) = v \in X, \quad u'(0) = 0.$

Fractional evolution equation:

$$D_t^{\delta}(u(t) - u(0)) = Au(t), \quad t > 0. \quad (11)$$

Initial conditions:

$$\begin{aligned} &u(0) = v \in X \text{ for } \ \delta \in (0,1]; \\ &u(0) = v \in X, \ u'(0) = 0 \text{ for } \ \delta \in (1,2]. \end{aligned} \quad (12)$$

$S_\delta(t)$ - solution operator to problem (11)-(12).

Subordination

Theorem: Assume the Cauchy problem (11) is well posed for some δ, $0 < \delta \le 2$, and admits a bounded solution operator $S_\delta(t)$. Let

$$g(s) \in \mathcal{CBF}^\delta, \quad s > 0. \tag{13}$$

Then the fractional Jeffreys-type equation (10) admits a bounded solution operator $S(t)$, such that

$$S(t) = \int_0^\infty \Phi(t,\tau) S_\delta(\tau)\, d\tau, \quad t > 0, \tag{14}$$

where $\Phi(t,\tau)$ is defined via the Laplace transform

$$\widehat{\Phi}(s,\tau) = \int_0^\infty e^{-st}\Phi(t,\tau)\, dt = \frac{g(s)^{1/\delta}}{s} \exp\left(-\tau g(s)^{1/\delta}\right)$$

and

$$\Phi(t,\tau) \ge 0, \quad \int_0^\infty \Phi(t,\tau)\, d\tau = 1, \qquad t,\tau > 0.$$

Injectivity

Theorem: Assume $f(t)$, $t \geq 0$, is a function with values in the Banach space X, such that $\|f(t)\| \leq C$, $t \geq 0$. Then the integral

$$\widetilde{f}(t) = \int_0^\infty \Phi(t,\tau) f(\tau)\, d\tau, \quad t > 0,$$

is well defined. Let $\widetilde{f}(t) = 0$ for a.e. $t \in (0,T)$, where $T > 0$. Then $f \equiv 0$.

This injectivity property is useful for the proof of uniqueness in some inverse problems.

Reference:

E. Bazhlekova, Subordination approach to multi-term time-fractional Jeffreys equation (in preparation).