Trigonometric identities: from Ptolemy to this day

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Hermite's cotangent identity

Theorem A (Hermite, 1872)

Let $a_1, a_2, \dots, a_n$ be complex numbers, no two of which differ by an integer multiple of π. Set

$$A_{n,k} = \prod_{\substack{1 \le j \le n \\ j \ne k}} \cot(a_k - a_j),$$

where the empty product for $A_{1,1} := 1$. Then

$$\prod_{j=1}^{n} \cot(z - a_j) = \cos\left(\frac{n\pi}{2}\right) + \sum_{k=1}^{n} A_{n,k} \cot(z - a_k)$$

Hermite's sine identity (generalized Ptolemy)

Hermite found another amazing result (generalizing a particular case presented by Glaisher without proof earlier)

$$\sum_{k=1}^{n} \frac{\prod_{j=1}^{n} \sin(a_j - b_k)}{\prod_{\substack{j=1 \\ j\neq k}}^{n} \sin(b_j - b_k)} = \sin(a_1 + \cdots + a_n - b_1 - \cdots - b_n),$$

When $n = 2$ this identity reduces to the trigonometric from of Ptolemy's theorem: for a quadrilateral inscribed in a circle the product of the lengths of its diagonals is equal to the sum of the products of the lengths of the pairs of is opposite sides.

Hermite presented an argument to prove this formula in a note published in 1885 which is independent of his another result also first published in 1872.

Hermite's rational trigonometric expansion

Theorem B (Hermite, 1872)

Let $a_1, a_2, \ldots, a_n$ be complex numbers, no two of which differ by an integer multiple of π. Suppose, further

$$F(\sin z, \cos z) = \sum_{j=0}^{n-1} t_j (\sin z)^j (\cos z)^{n-1-j}$$

is a homogeneous trigonometric polynomial of degree $n-1$. Then

$$\frac{F(\sin z, \cos z)}{\prod_{j=1}^{n} \sin(z - a_j)} = \sum \frac{F(\sin a_k, \cos a_k)}{\sin(z - a_k) \prod_{\substack{1 \le j \le n \\ j \neq k}} \sin(a_k - a_j)}$$

The sine identity would follow from degree n case of the polynomial F in the above identity.

Johnson's paper

In a nice paper *Trigonometric identities á la Hermite*, published in 2010 by the American Mathematical Monthly, Warren Johnson generalized Hermite's Theorem B to the case when $F(\sin(z), \cos(z))$ in

$$\frac{F(\sin z, \cos z)}{\prod_{j=1}^{n} \sin(z - a_j)}$$

is the trigonometric polynomial

$$F(\sin z, \cos z) = \sum_{\substack{r+s \le n \\ r,s \ge 0}} t_{r,s} (\sin(z))^r (\cos(z))^s$$

which, unlike Hermite's case, is (a) not necessarily homogeneous; (b) may have degree up to n. In particular, taking

$$F(\sin(z), \cos(z)) = \sin(z - b_1) \cdots \sin(z - b_n)$$

he got the Hermite sine identity (generalized Ptolemy).

Chu's paper

Two years before appearance of Johnson's paper Wenchang Chu published in 2008 a short note in AMS Proceedings entitled *Partial fraction decompositions and trigonometric sum identities*. Chu considers essentially the same problem just using a slightly more general form of the initial ratio, namely

$$\frac{P(e^{iz})}{\prod_{j=1}^{n}\sin(z-a_j)},$$

with $P(w)$ being a Laurent polynomial in w consisting of the terms w^k with $|k| \leq n$. Chu found a partial fraction decomposition of this ratio and showed that Hermite's cotangent and sine identities as well as many other can be obtained by a clever choice of P. One remarkable aspect of those two papers is that not only Johnson missed Chu's paper, but their references have empty intersection! In particular, Chu attributes Hermite's sine identity to Robert Gustafson...

Meijer, Nørlund and Braaksma

Another remarkable fact: both authors missed the works of Meijer (1940), Nørlund (1955) and Braaksma (1964), where they rediscovered and further generalized Hermite's sine identity in the context of analytic continuation of Mellin-Barnes integrals (aka Meijer's G and Fox's H functions). The connection is via Euler's reflection formula

$$\sin(\pi z) = \frac{\pi}{\Gamma(z)\Gamma(1-z)},$$

where Γ stands for Euler's gamma function. These implies that any trigonometric identity can be rewritten as a Γ identity.

Our work: motivation

In our paper *[D.Karp and E.Prilepkina, Hypergeometric differential equation and new identities for the coefficients of Nørlund and Bühring, SIGMA 12 (2016)]* we rediscovered Hermite's sine identity (in its Γ-form) and in *[D.B.Karp and E.G.Prilepkina, On Meijer's G function $G^{m,n}_{p,p}$ for $m+n=p$, Integral Transforms and Special Functions, Volume 34, 2023]* we applied it to derive simple and explicit analytic continuation formulas for Meijer's G function. This motivated me to look deeper into such identities (with the goal to get analytic continuation in more general cases) which lead to the recent paper *[A.Dyachenko and D.Karp, Trigonometric identities: from Hermite via Meijer, Nørlund and Braaksma to Chu and Johnson and beyond, Boletín de la Sociedad Matemática Mexicana, Volume 29, 2023]*. Our approach is a generalization of that of Chu and we allow arbitrary degrees of the numerator and denominator. Our general formulas are cumbersome so I only exhibit some corollaries.

Our work: selected corollaries

Let $r, n \in \mathbb{N}_0$, $\varkappa = r - n$ and $\lambda = \lfloor \varkappa/2 \rfloor$. Denote:
$\sin(z - \mathbf{a}) = \sin(z - a_0) \cdots \sin(z - a_r)$, $\mathcal{A} = \sum_{k=0}^{r} a_k$,
$\sin(z - \mathbf{b}) = \sin(z - b_0) \cdots \sin(z - b_n)$, $\mathcal{B} = \sum_{k=0}^{n} b_k$. For $\varkappa$ odd:

$$\frac{\sin(z - \mathbf{a})}{\sin(z - \mathbf{b})} = \sum_{t=0}^{\lambda} \mathcal{F}_{\varkappa-2t}(z) + \sum_{k=0}^{n} \frac{\sin(b_k - \mathbf{a})}{\sin(b_k - \mathbf{b}_{[k]}) \cdot \sin(z - b_k)},$$

where the first sum is non-void only for $\varkappa > 0$, and its terms are

$$\mathcal{F}_{\varkappa-2t}(z) = \sum_{m=0}^{t} \frac{(-1)^{\lambda+t-m}}{2^{\varkappa-1}} \times$$

$$\sum_{\substack{0 \le j_1 < \cdots < j_{t-m} \le r \\ 0 \le k_1 \le \cdots \le k_m \le n}} \sin\Big((\varkappa - 2t)z + \mathcal{B} - \mathcal{A} + 2(b_{k_1} + \cdots + b_{k_m} + a_{j_1} + \cdots + a_{j_{t-m}})\Big).$$

The symbol $\mathbf{b}_{[k]}$ signifies vector $\mathbf{b}$ without k-th component.

Our work: selected corollaries

If $\varkappa$ is even, then

$$\frac{\sin(z-\mathbf{a})}{\sin(z-\mathbf{b})} = \sum_{t=0}^{\lambda} \mathcal{F}_{\varkappa-2t}(z) + \sum_{k=0}^{n} \frac{\sin(b_k-\mathbf{a})}{\sin(b_k-\mathbf{b}_{[k]})} \cot(z-b_k),$$

where (note that $\mathcal{F}_0(z)=0$ if $\lambda<0$)

$$\mathcal{F}_0 = \sum_{m=0}^{\lambda} \frac{(-1)^m}{2^{\varkappa}} \times$$

$$= \sum_{\substack{0\le j_1<\cdots<j_{\lambda-m}\le r \\ 0\le k_1\le\cdots\le k_m\le n}} \cos\left(\mathcal{B}-\mathcal{A}+2(b_{k_1}+\cdots+b_{k_m}+a_{j_1}+\cdots+a_{j_{\lambda-m}})\right),$$

and, for $t=0,1,\ldots,\lambda-1$,

$$\mathcal{F}_{\varkappa-2t}(z) = \sum_{m=0}^{t} \frac{(-1)^{\lambda+t-m}}{2^{\varkappa-1}} \times$$

$$\sum_{\substack{0\le j_1<\cdots<j_{t-m}\le r \\ 0\le k_1\le\cdots\le k_m\le n}} \cos\Big((\varkappa-2t)z+\mathcal{B}-\mathcal{A}+2(b_{k_1}+\cdots+b_{k_m}+a_{j_1}+\cdots+a_{j_{t-m}})\Big).$$

Our work: generalized Hermite's sine identity

Moreover, for even $\varkappa = 2\lambda$ we additionally have

$$\sum_{k=0}^{n} \frac{\sin(b_k - \mathbf{a})}{\sin(b_k - \mathbf{b}_{[k]})} = \sum_{m=0}^{\lambda} \frac{(-1)^m}{4^\lambda} \times$$

$$\sum_{\substack{0 \le j_1 < \cdots < j_{\lambda-m} \le n+2\lambda \\ 0 \le k_1 \le \cdots \le k_m \le n}} \sin\left(\mathcal{B} - \mathcal{A} + 2(b_{k_1} + \cdots + b_{k_m} + a_{j_1} + \cdots + a_{j_{t-m}})\right).$$

Reminder: $\mathcal{A} = \sum_{k=0}^{r} a_k$, $\mathcal{B} = \sum_{k=0}^{n} b_k$,

$\mathbf{b}_{[k]}$ signifies vector $\mathbf{b}$ without k-th component.

Example 1: $r = n + 1$

If $r = n + 1$ (vector $\mathbf{a}$ is longer than $\mathbf{b}$ by 1 component, $\varkappa = 1$)

$$\frac{\sin(z-\mathbf{a})}{\sin(z-\mathbf{b})} = \sin(z + \mathcal{B} - \mathcal{A}) + \sum_{k=0}^{n} \frac{\sin(b_k - \mathbf{a})}{\sin(z - b_k)\sin(b_k - \mathbf{b}_{[k]})}.$$

Using the fact that sum of residues of a rational function over the Riemann sphere is zero, we obtain the following exotic identity from the above

$$4\sum_{k=0}^{n} \frac{e^{i(\nu+b_k)}\sin(b_k - \mathbf{a})}{\sin(b_k - \mathbf{b}_{[k]})} = \sum_{k=0}^{n+1} e^{2ia_k} - \sum_{k=0}^{n} e^{2ib_k} - e^{2i\nu},$$

where $\nu = \mathcal{A} - \mathcal{B}$. Separating real and imaginary parts of the above identity we can get similar formulas with cosines and sines in place of exponentials.

Example 2: $r = n+2$

If $r = n+2$ (vector $\mathbf{a}$ is longer than $\mathbf{b}$ by 2 components, $\varkappa = 2$)

$$\frac{\sin(z-\mathbf{a})}{\sin(z-\mathbf{b})} = -\frac{1}{2}\cos(\mathcal{B}-\mathcal{A}+2z)$$

$$+\frac{e^{i(\mathcal{B}-\mathcal{A})}}{4}\left(\sum_{k=0}^{n+2} e^{2ia_k} - \sum_{k=0}^{n} e^{2ib_k}\right) + \sum_{k=0}^{n} \frac{e^{i(b_k-z)}\sin(b_k-\mathbf{a})}{\sin(z-b_k)\sin(b_k-\mathbf{b}_{[k]})}.$$

Separating the real and imaginary parts leads to the identities:

$$\frac{\sin(z-\mathbf{a})}{\sin(z-\mathbf{b})} = -\frac{1}{2}\cos(\mathcal{B}-\mathcal{A}+2z) + \frac{1}{4}\sum_{k=0}^{n+2}\cos(\mathcal{B}-\mathcal{A}+2a_k)$$

$$-\frac{1}{4}\sum_{k=0}^{n}\cos(\mathcal{B}-\mathcal{A}+2b_k) + \sum_{k=0}^{n}\frac{\sin(b_k-\mathbf{a})}{\sin(b_k-\mathbf{b}_{[k]})}\cot(z-b_k)$$

and (generalized Hermite's sine identity)

$$4\sum_{k=0}^{n}\frac{\sin(b_k-\mathbf{a})}{\sin(b_k-\mathbf{b}_{[k]})} = \sum_{k=0}^{n+2}\sin(\mathcal{B}-\mathcal{A}+2a_k) - \sum_{k=0}^{n}\sin(\mathcal{B}-\mathcal{A}+2b_k).$$

Gosper's *Experiments in q-trigonometry (2001)*

Following William Gosper define for $0 < q < 1$

$$\sin_q(\pi z) = q^{(z-1/2)^2} \prod_{n=1}^{\infty} \frac{(1-q^{2n-2z})(1-q^{2n+2z-2})}{(1-q^{2n-1})^2} = \frac{\vartheta_1(z,p)}{\vartheta_1(\pi/2,p)},$$

$$\cos_q(\pi z) = q^{z^2} \prod_{n=1}^{\infty} \frac{(1-q^{2n-2z-1})(1-q^{2n+2z-1})}{(1-q^{2n-1})^2} = \frac{\vartheta_1(z+\pi/2,p)}{\vartheta_1(\pi/2,p)}$$

where $\ln(p) \cdot \ln(q) = \pi^2$ and Jacobi's theta is defined by

$$\vartheta_1(z,p) = 2\sum_{n=0}^{\infty} (-1)^n p^{(n+1/2)^2} \sin[(2n+1)z].$$

Define $\mathcal{C} = \sum_{j=1}^{n} c_j$, then it is conjectured that

$$\sin_q(\mathcal{C}) \prod_{i=1}^{n} \frac{\sin_q(a_i - c_i)}{\sin_q(a_i)}$$

$$= \sum_{i=1}^{n} \frac{\sin_q(c_i)\sin_q(a_i - \mathcal{C})}{\sin_q(a_i)} \prod_{1 \le j \le n}^{j \neq i} \frac{\sin_q(c_j + a_i - a_j)}{\sin_q(a_i - a_j)}.$$

Possible generalization: double sine function

The double sine function S_2 is defined for $\boldsymbol{\omega} = (\omega_1, \omega_2)$, $\omega_i > 0$, by the formula

$$S_2(x, \boldsymbol{\omega}) = \frac{\prod_{\mathbf{n} \geq 0}(\mathbf{n} \cdot \boldsymbol{\omega} + x)}{\prod_{\mathbf{m} \geq 1}(\mathbf{m} \cdot \boldsymbol{\omega} - x)} = \frac{\Gamma_2(|\boldsymbol{\omega}| - x, \boldsymbol{\omega})}{\Gamma_2(x, \boldsymbol{\omega})},$$

where $\mathbf{n} = (n_1, n_2)$, $\mathbf{m} = (m_1, m_2)$, $|\boldsymbol{\omega}| = \omega_1 + \omega_2$, and $\Gamma_2(x, \boldsymbol{\omega})$ is the double gamma function. The double sine is symmetric, $S_2(x, (\omega_1, \omega_2)) = S_2(x, (\omega_2, \omega_1))$ and has two pseudo-periods:

$$S_2(x + \omega_1, \boldsymbol{\omega}) = \frac{S_2(x, \boldsymbol{\omega})}{S_1(x, \omega_2)} = \frac{S_2(x, \boldsymbol{\omega})}{2 \sin(\pi x / \omega_2)}$$

and, similarly,

$$S_2(x + \omega_2, \boldsymbol{\omega}) = \frac{S_2(x, \boldsymbol{\omega})}{S_1(x, \omega_1)} = \frac{S_2(x, \boldsymbol{\omega})}{2 \sin(\pi x / \omega_1)}.$$

БЛАГОДАРЯ ВИ ЗА ВНИМАНИЕТО!

THANK YOU FOR ATTENTION!