ANNUAL SESSION - AGT Section
4 December 2023

FROM THE LE ROY FUNCTIONS – TO GENERALIZATIONS OF THE WRIGHT-FOX HYPERGEOMETRIC FUNCTIONS AND FOX *H*-FUNCTIONS

V. Kiryakova and J. Paneva-Konovska

In Memory of Prof. Ivan Dimovski (1934–2023)

mathematics

MDPI

Review

A Guide to Special Functions in Fractional Calculus

Virginia Kiryakova

2021, https://doi.org/math9010106

Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, 1113 Sofia, Bulgaria;
virginia@diogenes.bg

Abstract: *Dedicated to the memory of Professor Richard Askey (1933–2019) and to pay tribute to the Bateman Project*. Harry Bateman planned his "shoe-boxes" project (accomplished after his death as *Higher Transcendental Functions*, Vols. 1–3, 1953–1955, under the editorship by A. Erdélyi) as a "*Guide to the Functions*". This inspired the author to use the modified title of the present survey. Most of the standard (classical) Special Functions are representable in terms of the Meijer G-function and, specially, of the generalized hypergeometric functions ${}_pF_q$. These appeared as solutions of differential equations in mathematical physics and other applied sciences that are of integer order, usually of second order. However, recently, mathematical models of fractional order are preferred because they reflect more adequately the nature and various social events, and these needs attracted attention to *"new" classes* of special functions as their solutions, the so-called *Special Functions of Fractional Calculus (SF of FC)*. Generally, under this notion, we have in mind the Fox H-functions, their most widely used cases of the Wright generalized hypergeometric functions ${}_p\Psi_q$ and, in particular, the Mittag–Leffler type functions, among them the "Queen function of fractional calculus", the Mittag–Leffler function. These fractional indices/parameters extensions of the classical special functions became an unavoidable tool when fractalized models of phenomena and events are treated. Here, we try to review some of the basic results on the theory of the SF of FC, obtained in the author's works for more than 30 years, and support the wide spreading and important role of these functions by several examples.

Keywords: special functions; generalized hypergeometric functions; fractional calculus operators; integral transforms

MSC: 33C60; 33E12; 26A33; 44A20

Citation: Kiryakova, V. A Guide to Special Functions in Fractional Calculus. *Mathematics* **2021**, *9*, 106. https://doi.org/10.3390/math9010106

Received: 15 December 2020
Accepted: 24 December 2020
Published: 5 January 2021

1. Historical Introduction

Special functions are particular mathematical functions that have more or less established names and notations due to their importance in mathematical analysis, functional analysis, geometry, physics, astronomy, statistics or other applications (Wikipedia: Special Functions [1]). It might be Euler, who started to talk, since 1720, about lots of the standard special functions. He defined the Gamma-function as a continuation of the factorial, also the Bessel functions and looked after the elliptic functions. Several (theoretical and applied) scientists started to use such functions, introduced their notations and named them after famous contributors. Thus, the notions as the Bessel and cylindrical functions; the Gauss, Kummer, Tricomi, confluent and generalized hypergeometric functions; the classical orthogonal polynomials (as Laguerre, Jacobi, Gegenbauer, Legendre, Tchebisheff, Hermite, etc.); the incomplete Gamma- and Beta-functions; and the Error functions, the Airy, Whittaker, etc. functions appeared and a long list of handbooks on the so-called "*Special Functions of Mathematical Physics*" or "*Named Functions*" (we call them also "*Classical Special Functions*") were published. We mention only some of them in this survey.

As Richard Askey (*to whose memory we dedicate this survey*) confessed in his lectures [2] on orthogonal polynomials and special functions: "Now, there are relatively large num-

Before to present the considered new classes of special functions, let us remind the evolution in the ideas about some particular cases, called by now as "SF of FC":

$$\exp(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k+1)} \longrightarrow \text{(M-L) } E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}$$

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)} \longrightarrow \text{(Prabh.) } E_{\alpha,\beta}^{\tau}(z)=\sum_{k=0}^{\infty}\frac{(\tau)_k}{\Gamma(\alpha k+\beta)}\frac{z^k}{k!}$$

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)} \longrightarrow$$

$$\text{(multi-ML: VK) } E_{(\alpha_i),(\beta_i)}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha_1 k+\beta_1)\ldots\Gamma(\alpha_m k+\beta_m)}$$

$$\text{(Prabh.) } E_{\alpha,\beta}^{\tau}(z)=\sum_{k=0}^{\infty}\frac{(\tau)_k}{\Gamma(\alpha k+\beta)}\frac{z^k}{k!} \longrightarrow$$

$$\text{(JPK) } E_{(\alpha_i),(\beta_i)}^{(\tau_i),m}(z)=\sum_{k=0}^{\infty}\frac{(\tau_1)_k\ldots(\tau_m)_k}{\Gamma(\alpha_1 k+\beta_1)\ldots\Gamma(\alpha_m k+\beta_m)}\frac{z^k}{(k!)^m}$$

All these SF fall in the scheme of the Wright (Wright-Fox) generalized hypergeometric functions

$$ {}_p\Psi_q\left[\begin{array}{c}(a_1,A_1),...,(a_p,A_p)\\(b_1,B_1),...,(b_q,B_q)\end{array}\Bigg| z\right]=\sum_{k=0}^{\infty}\frac{\Gamma(a_1+kA_1)\dots\Gamma(a_p+kA_p)}{\Gamma(b_1+kB_1)\dots\Gamma(b_q+kB_q)}\,\frac{z^k}{k!} $$

$$ = H^{1,p}_{p,q+1}\left[-z\,\Bigg|\,\begin{array}{c}(1-a_1,A_1),\dots,(1-a_p,A_p)\\(0,1),(1-b_1,B_1),\dots,(1-b_q,B_q)\end{array}\right]. \qquad (1) $$

Denote: $\rho=\prod\limits_{i=1}^{p}A_i^{-A_i}\prod\limits_{j=1}^{q}B_j^{B_j}$, $\Delta=\sum\limits_{k=1}^{j}B_j-\sum\limits_{i=1}^{p}A_i$. If $\Delta>-1$, the ${}_p\Psi_q$-function is an entire function of $z\in\mathbb{C}$, but if $\Delta=-1$, the series is absolutely convergent in the disk $\{|z|<\rho\}$, while ....

If all $A_1=\dots=A_p=1, B_1=\dots=B_q=1$, the Wright g.h.f. reduces to the *generalized hypergeometric ${}_pF_q$-function* which itself is a case of the Meijer G-function,

$$ {}_p\Psi_q\left[\begin{array}{c}(a_1,1),\dots,(a_p,1)\\(b_1,1),\dots,(b_q,1)\end{array}\Bigg| z\right]=c\;{}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;\,z) $$

$$ =\sum_{k=0}^{\infty}\frac{(a_1)_k\dots(a_p)_k}{(b_1)_k\dots(b_q)_k}\,\frac{z^k}{k!}=G^{1,p}_{p,q+1}\left[-z\,\Bigg|\,\begin{array}{c}1-a_1,\dots,1-a_p\\0,1-b_1,\dots,1-b_q\end{array}\right];\; c=\dots $$

Definition. (Ch. Fox (1961) The Fox H-function is a generalized hypergeometric function, defined by means of the Mellin-Barnes type contour integral

$$H_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix} (a_i, A_i)_1^p \\ (b_j, B_j)_1^q \end{matrix}\right] = \frac{1}{2\pi i}\int_{\mathcal{L}} \mathcal{H}_{p,q}^{m,n}(s)\, z^{-s} ds, \quad \text{with} \tag{2}$$

$$\mathcal{H}_{p,q}^{m,n}(s) = \frac{\prod_{j=1}^{m} \Gamma(b_j + B_j s) \prod_{i=1}^{n} \Gamma(1 - a_i - A_i s)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j - B_j s) \prod_{i=n+1}^{p} \Gamma(a_i + A_i s)},$$

with complex variable $z \neq 0$ and a contour $\mathcal{L}$ (3 types) in the complex domain; the orders (m, n, p, q) are non negative integers so that $0 \leq m \leq q$, $0 \leq n \leq p$, the parameters $A_i > 0, B_j > 0$ are positive, and a_i, b_j, $i = 1, ..., p; j = 1, ..., q$ are arbitrary complex such that $A_i(b_j + l) \neq B_j(a_i - l' - 1)$, $l, l' = 0, 1, 2, ...;$ $i = 1, ..., n;$ $j = 1, ..., m$. For details on types of contours $\mathcal{L}$ and the properties of the H-function, see in many contemporary handbooks on SF as ..., where its behaviour is described in term of the parameters: ρ, Δ, ∇, μ, ... For $\forall A_i = B_j = 1$, the H-function reduces to a G-function.

However, $\exp(z)=\sum_{k=0}^{\infty}\frac{z^k}{k!} \longrightarrow$ (Le Roy) $F_\gamma(z)=\sum_{k=0}^{\infty}\frac{z^k}{(k!)^\gamma}$

$\longrightarrow$ (MLR: Gerhold, Garra-Polito) $F_{\alpha,\beta}^{(\gamma)}(z)=\sum_{k=0}^{\infty}\frac{z^k}{[\Gamma(\alpha k+\beta)]^\gamma}$

($\gamma>0$) $\longrightarrow$ (Rogosin) $F_{(\alpha,\beta)_m}^{(\gamma)_m}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\prod_{j=1}^{m}[\Gamma(\alpha_j k+\beta_j)]^{\gamma_j}}$.

Then, from the $3m$-multi-M-L and $3m$-MLR functions, we went further to a new class of SF, as follows:

Definition. Multi-index Mittag-Leffler-Prabhakar functions of Le Roy type (multi-MLPR), suppose all $4m$ parameters are >0:

$$\mathbb{F}_m(z):=\mathbb{F}_{\alpha_i,\beta_i;\tau_i}^{\gamma_i;m}(z) \tag{3}$$

$$=\sum_{k=0}^{\infty}\frac{(\tau_1)_k\ldots(\tau_m)_k}{(k!)^m}\cdot\frac{z^k}{[\Gamma(\alpha_1 k+\beta_1)]^{\gamma_1}\ldots[\Gamma(\alpha_m k+\beta_m)]^{\gamma_m}}$$

$$=\sum_{k=0}^{\infty}c_k\,z^k,\quad\text{with}\quad c_k=\prod_{i=1}^{m}\left\{\frac{\Gamma(k+\tau_i)}{\Gamma(k+1)}\cdot\frac{1}{\Gamma(\tau_i)}\cdot\frac{1}{[\Gamma(\alpha_i k+\beta_i)]^{\gamma_i}}\right\}.$$

Theorem. *Suppose* $\forall i = 1, ..., m : \alpha_i > 0, \beta_i > 0, \gamma_i > 0, \tau_i > 0$, *and* $\sum_{i=1}^{m} \alpha_i \gamma_i > 0$. *The multi-index MLPR-function* (3) *is an entire function of the complex variable* z *of order* ρ *and type* σ*:*

$$\rho = \frac{1}{\alpha_1\gamma_1 + \cdots + \alpha_m\gamma_m}, \ \textit{i.e.} \ \frac{1}{\rho} = \alpha_1\gamma_1 + \cdots + \alpha_m\gamma_m, \tag{4}$$

and

$$\sigma = \frac{1}{\rho}\left(\prod_{i=1}^{m}(\alpha_i)^{-\alpha_i\gamma_i}\right)^{\rho},$$

that is,

$$\sigma = \frac{\alpha_1\gamma_1 + \cdots + \alpha_m\gamma_m}{\left(\alpha_1^{\alpha_1\gamma_1}\cdots\alpha_m^{\alpha_m\gamma_m}\right)^{1/(\alpha_1\gamma_1+\cdots+\alpha_m\gamma_m)}}. \tag{5}$$

Some of our results include: - Mellin-Barnes type integral representations for $\mathbb{F}_m(z)$; - Laplace transform of this function; - Image of $\mathbb{F}_m(z)$ under operators of FC (Erdélyi-Kober fract. integrals), etc.

In all cases the results appear again as functions of same multi-index MLPR-type but with increased multiplicity m and additional parameters.

Published in 2 papers (2023): VK + JPK + Rogosin/ Dub.

Definition. The so-called I-function was defined by Rathie, 1997, by means of a kind of Mellin-Barnes type integral

$$I_{p,q}^{m,n}\left[z\,\middle|\,\begin{array}{l}(a_i,A_i,\alpha_i)_1^p\\(b_j,B_j,\beta_j)_1^q\end{array}\right]=\frac{1}{2\pi i}\int_{\mathcal{L}}\mathcal{I}_{p,q}^{m,n}(s)\,z^{-s}ds,\quad z\neq 0,$$

$$\text{with}\quad \mathcal{I}_{p,q}^{m,n}(s)=\frac{\prod\limits_{j=1}^{m}\Gamma^{\beta_j}(b_j+B_js)\prod\limits_{i=1}^{n}\Gamma^{\alpha_i}(1-a_i-A_is)}{\prod\limits_{j=m+1}^{q}\Gamma^{\beta_j}(1-b_j-B_js)\prod\limits_{i=n+1}^{p}\Gamma^{\alpha_i}(a_i+A_is)}. \tag{6}$$

Note that if $\forall\alpha_i=\forall\beta_j=1$, $i=1,\ldots,p$, $j=1,\ldots,q$, this is the Fox H-function. But in general, these are NOT positive integers. Then, we have a new multi-valued function with singularities that are branch points, etc. Some more simple case of this SF, is the $\bar{H}$-function of Inayat-Hussain, 1987, where in particular some of the α_i,β_j are equal to 1, namely: $\alpha_i=1, i=n+1,\ldots,p$ and $\beta_j=1, j=1,\ldots,m$:

$$\overline{H}_{p,q}^{m,n}\left[z\,\middle|\,\begin{array}{l}(a_i,A_i,\alpha_i)_1^p\\(b_j,B_j,\beta_j)_1^q\end{array}\right]=\frac{1}{2\pi i}\int_{\mathcal{L}}\frac{\prod\limits_{j=1}^{m}\Gamma^{\mathbf{1}}(b_j+B_js)\prod\limits_{i=1}^{n}\Gamma^{\alpha_i}(1-a_i-A_is)}{\prod\limits_{j=m+1}^{q}\Gamma^{\beta_j}(1-b_j-B_js)\prod\limits_{i=n+1}^{p}\Gamma^{\mathbf{1}}(a_i+A_is)}z^{-s}ds.$$

It happens that some important SF that are not H-functions and ${}_p\Psi_q$-functions, can be presented in terms of the I- and $\bar{H}$-functions. This was an argument in the initial works by Inayat-Hussain (with title: "... hypergeometric series derivable from Feynman integrals") and by Rathie, and we stuck on some hints that other more popular SF fall in the scheme of the I-functions, as: the polylogaritm function, the Riemann ζ-function, Mathiew series, etc. Just for example, the polylogarithm function

$$\mathrm{Li}_\alpha(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^\alpha}, \quad |z| < 1, \ \alpha \in \mathbb{C},$$

can be identified as such a function. Namely, we derived a Mellin-Barnes type integral representation as for a $\bar{H}$-function:

$$\mathrm{Li}_\alpha(z) = -\frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} \frac{\Gamma^{\alpha+1}(s)\,\Gamma(1-s)}{\Gamma^\alpha(1+s)} (-z)^s \, ds$$

$$= -\overline{H}_{1,2}^{1,1}\left[-z \,\middle|\, \begin{array}{c} (1,1,\alpha+1) \\ (1,1,1),(0,1,\alpha) \end{array}\right], \ \alpha > 0, \ \mathcal{L} = \{c-i\infty, c+i\infty\}.$$

(All singularities of the Gamma's in numerator are to the left of $s=0$ and to the right of $s=1$, and one can take $c = 1/2$.)

The generalized Hurwitz-Lerch Zeta function:

$$\Phi_{\lambda,\nu,\mu}^{(\rho,\sigma,\kappa)}(z,s,a) = \sum_{n=0}^{\infty} \frac{(\lambda)_{\rho n}(\mu)_{\sigma n}}{(\nu)_{\kappa n}\, n!} \cdot \frac{z^n}{(n+a)^s}, \quad |z| < \rho^*. \qquad (7)$$

According to (Srivastava-Saxena-Pogány-Saxena, 2011), it has the following contour integral representation:

$$\Phi_{\lambda,\nu,\mu}^{(\rho,\sigma,\kappa)}(z,s,a) = \frac{\Gamma(\nu)}{\Gamma(\lambda)\Gamma(\mu)}$$

$$\times \int_{\mathcal{L}} \frac{\Gamma(-\zeta)\Gamma(\lambda+\rho\zeta)\Gamma(\mu+\sigma\zeta)\,\Gamma^s(\zeta+a)}{\Gamma(\nu+\kappa\zeta)\,\Gamma^s(\zeta+a+1)} (-z)^{\zeta}\, d\zeta, \qquad (8)$$

for $|\arg(-z)| < \pi$, and path of integration $\mathcal{L} = (c - i\infty, c + i\infty)$ that separates the poles of $\Gamma(-\zeta)$, $\Gamma(\lambda+\rho\zeta)$, $\Gamma(\mu+\sigma\zeta)$, $\Gamma(\zeta+a)$.
Then, the relation with the $\bar{H}$-function is obtained:

$$\Phi_{\lambda,\nu,\mu}^{(\rho,\sigma,\kappa)}(z,s,a) = \frac{\Gamma(\nu)}{\Gamma(\lambda)\Gamma(\mu)}$$

$$\times\ \overline{H}_{3,3}^{1,3}\left[-z \,\middle|\, \begin{array}{c} (1-\lambda,\rho,1),(1-\mu,\sigma,1),(1-a,1,s) \\ (0,1),(1-\nu,\kappa,1),(-a,1,s) \end{array}\right]. \qquad (9)$$

Several important special cases are considered, incl. Riemann Zeta function $\zeta(s) = \sum_0^{\infty} z^n/n^s$ $(z=1,\ a=0,\ \ldots)$, etc.

Our hypothesis was - now proved, that under suitable conditions on the contour and situation of the singularities of the Gamma-functions, the Le Roy type functions can also be represented in terms of I- and in particular, $\bar{H}$-functions (and of almost same kind as the polylogarithm!). Namely:

$$F^{(\gamma)}(z) = I^{1,1}_{1,2}\left[-z\left|\begin{array}{c}(0,1,1)\\(0,1,1),(0,1,\gamma)\end{array}\right.\right], \text{ the original Le Roy function;}$$

$$F^{(\gamma)}_{\alpha,\beta}(z) = I^{1,1}_{1,2}\left[-z\left|\begin{array}{c}(0,1,1)\\(0,1,1),(1-\beta,\alpha,\gamma)\end{array}\right.\right], \text{ Gerhold, Garra-Polito,...;}$$

$$F^{(\gamma)}_{\alpha,\beta,\tau}(z) = I^{1,1}_{1,2}\left[-z\left|\begin{array}{c}(1-\tau,1,1)\\(0,1,1),(1-\beta,\alpha,\gamma)\end{array}\right.\right], \text{ Prabhakar type, JPK.}$$

While, for the multi-index M-L function of Le Roy type (with $\forall\tau_i = 1$, as in Rogosin),

$$F^{(\gamma_i)_1^m}_{(\alpha_i)_1^m,(\beta_i)_1^m}(z) = I^{1,1}_{1,m+1}\left[-z\left|\begin{array}{c}(0,1,1)\\(0,1,1),(1-\beta_i,\alpha_i,\gamma_i)_1^m\end{array}\right.\right] = \overline{H}^{1,1}_{1,m+1}(-z).$$

but: $E_{(\alpha_i)_1^m,(\beta_i)_1^m}(z) = H^{1,1}_{1,m+1}\left[-z\left|\begin{array}{c}(0,1)\\(0,1),(1-\beta_i,\alpha_i)_1^m\end{array}\right.\right]$, VK, H-f.

Before to present our result in the most general case of Le Roy type functions $\mathbb{F}_m$ (with $4m$ parameters), it is interesting to introduce also a *"fractional powers" extension of the Wright (Fox-Wright) functions* ${}_p\Psi_q$.

Theorem. *Define the generalized Wright-Fox function:*

$$
{}_p\widetilde{\Psi}_q \left[\begin{array}{c} (a_j^*, A_j^*; \alpha_j^*)_{j=1}^p \\ (b_i^*, B_i^*; \beta_i^*)_{i=1}^q \end{array} \middle| z \right] := \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma^{\alpha_j^*}(A_j^* k + a_j^*)}{\prod_{i=1}^{q} \Gamma^{\beta_i^*}(B_i^* k + b_i^*)} \cdot \frac{z^k}{k!}
$$

$$
= \overline{H}^{1,p}_{p,q+1} \left[-z \middle| \begin{array}{c} (1-a_j^*, A_j, \alpha_j^*)_1^p \\ (0,1), (1-b_i^*, B_i^*, \beta_i^*)_1^q \end{array} \right]. \qquad (10)
$$

Here the parameters for this $\bar{H}$ function are:

$\mu = 1 + \sum_{i=1}^{q} \beta_i^* B_i^* - \sum_{j=1}^{p} \alpha_j^* A_j^*$ *and* $R = \prod_{i=1}^{q} (B_i^*)^{\beta_i^* B_i^*} / \prod_{j=1}^{p} (A_j^*)^{\alpha_j^* A_j^*}$.

Then this series is an entire function (i.e., abs. conv. for all $0<|z|<\infty$) if $\mu > 0$; or if $\mu = 0$: is analyic in $|z|<R$. Suitable cuts are to be inserted in $\mathbb{C}$ so to fix single-valued branches for z^k and for included multi-valued Γ-functions with arbitrary powers.

Theorem. *Under the assumptions*

$$\alpha_i > 0, \beta_i > 0, \gamma_i > 0, \tau_i > 0, \; \forall i = 1, ..., m, \;\; \sum_{i=1}^{m} \alpha_i \gamma_i > 0, \qquad (11)$$

the multi-index Mittag-Leffler-Prabhakar function of Le Roy type $\mathbb{F}_m$ *can be represented in terms of the generalized Wright-Fox function* ${}_p\widetilde{\Psi}_q$ *and more generally, as* $\overline{H}$*-function and* I*-function:*

$$\mathbb{F}_m(z) := \mathbb{F}^{\gamma_i;\, m}_{\alpha_i;\beta_i;\,\tau_i}(z) = T \cdot {}_m\widetilde{\Psi}_{2m-1}\left[\begin{array}{c} (\tau_i, 1, 1)_1^m \\ (1,1,1)_{(m-1)-times}, (\beta_i, \alpha_i, \gamma_i)_1^m \end{array} \middle| z \right]$$

$$= T \cdot \overline{H}^{1,m}_{m,2m}\left[-z \middle| \begin{array}{c} (1-\tau_i, 1, 1)_1^m \\ (0,1)_{m-times}, (1-\beta_i, \alpha_i, \gamma_i)_1^m \end{array} \right] \qquad (12)$$

$$= T \cdot I^{1,m}_{m,2m}\left[-z \middle| \begin{array}{c} (1-\tau_i, 1, 1)_1^m \\ (0,1,1)_{m-times}, (1-\beta_i, \alpha_i, \gamma_i)_1^m \end{array} \right], \quad T = 1/\prod_{i=1}^{\Gamma}(\tau_i).$$

Note: For these ${}_m\widetilde{\Psi}_{2m-1}$ and $\bar{H}$-functions: $\mu = \sum_{i=1}^m \alpha_i \gamma_i > 0$, OK. The singularities of the involved Γ-functions in the M-B type integral for $\bar{H}^{1,m}_{m,2m}$ lie resp. in intervals $s > 1$, $s < -m_0$, $s < -\widetilde{m}_0$, where $m_0 := \min\{\tau_1, ..., \tau_m\} > 0$, $\widetilde{m}_0 := \min\{\beta_1/\alpha_1, ..., \beta_m/\alpha_m\} > 0$. Then a contour $(c - i\infty, c + i\infty)$ with $c \in (-\min(m_0, \widetilde{m}_0), 1)$, will be in a strip with no branch points and no branch cuts inside.

Definition. (Gelfond-Leontiev, 1951) Let the function $\varphi(\lambda)=\sum_{k=0}^{\infty}\varphi_k\lambda^k$ be an entire function with a growth (order $\rho>0$ and type $\sigma\neq 0$), such that $\lim_{k\to\infty} k^{\frac{1}{\rho}}\sqrt[k]{|\varphi_k|}=(\sigma e\rho)^{\frac{1}{\rho}}$. Then the operation

$$f(z)=\sum_{k=0}^{\infty}a_k z^k \overset{D_\varphi}{\longmapsto} D_\varphi f(z)=\sum_{k=1}^{\infty}a_k\frac{\varphi_{k-1}}{\varphi_k}z^{k-1}, \tag{13}$$

is called a G-L operator of generalized differentiation with respect to the function $\varphi(\lambda)$, and the corresponding G-L operator of generalized integration can be also introduced:

$$L_\varphi f(z)=\sum_{k=0}^{\infty}a_k\frac{\varphi_{k+1}}{\varphi_k}z^{k+1}, \tag{14}$$

Evidently, $D_\varphi L_\varphi f(z)=f(z)$. It happens also that the function φ is eigen-function of the G-L operator generated by it.

The classical diff./integr. are generated by $\exp z$; while for the M-L function and multi-index M-L functions we constructed corresp. G-L operators, defined by series of the above forms. And specially, for the G-L gen. integr. we provided also fractional integrals' representations with kernels $H^{1,0}_{1,1}$, resp. $H^{m,0}_{m,m}$.

We aimed to construct G-L operators **D** and **L**, generated by the Le Roy type functions, in the case $F^{(\gamma)_m}_{(\alpha,\beta)_m}$, by means of their coefficients $\varphi_k = 1/\prod_{i=1}^{m} \Gamma^{\gamma_i}(\alpha_i k + \beta_i)$. And we proved the eigen-function relations:

$$\mathbf{D}\, F^{(\gamma)_m}_{(\alpha,\beta)_m}(\lambda z) = \lambda\, F^{(\gamma)_m}_{(\alpha,\beta)_m}(\lambda z), \quad \lambda \neq 0. \tag{15}$$

For the G-L integration the corresponding relation is

$$\mathbf{L}\, F^{(\gamma)_m}_{(\alpha,\beta)_m}(\lambda z) = \frac{1}{\lambda}\, F^{(\gamma)_m}_{(\alpha,\beta)_m}(\lambda z) - \frac{1}{\lambda \prod\limits_{i=1}^{m} \Gamma^{\gamma_i}(\beta_i)}, \quad \lambda \neq 0. \tag{16}$$

Theorem. *The G-L integration* **L**, *generated by means of the Le Roy type function* $F^{(\gamma)_m}_{(\alpha,\beta)_m}$ *as a series, can be represented also by means of the integral operator*

$$\mathbb{I}^m f(z) = \mathbf{L}_{mMLR}\, f(z) = z \int\limits_0^1 I^{m,0}_{m,m}\left[\sigma \,\middle|\, \begin{array}{c} (\beta_i, \alpha_i, \gamma_i)_1^m \\ (\beta_i - \alpha_i, \alpha_i, \gamma_i)_1^m \end{array}\right] f(z\sigma) d\sigma. \tag{17}$$

This can be interpreted (again) as a kind of a generalized fractional integration of multi-order $(\alpha_1, \ldots, \alpha_m)$. *Note that this kernel is well defined I-function in the unit disc and vanishes for* $|z| > 1$.

In the case $m = 1$ the operators $\mathbb{I}^1$ can be considered as analogues of the Erdélyi-Kober (E-K) operators ! Namely, we the following composition/decomposition property, like in the GFC, where the (m-tuple) gen. fr. integrals can be represented also as commutable compositions of m single E-K fractional integrals.

Theorem. *For entire functions $f(z)$,*

$$\mathbb{I}^m f(z) = \left[\prod_{i=1}^{m} \mathbb{I}_i^1\right] f(z) = \mathbb{I}_m^1 \left\{\mathbb{I}_{m-1}^1 \cdots \left[\mathbb{I}_1^1\right]\right\} f(z). \qquad (18)$$

The above composition is commutable. The analogues of the E-K fractional integrals of order $\alpha_i > 0$ have the form

$$\mathbb{I}_i^1 f(z) = z \int_0^1 I_{1,1}^{1,0} \left[\sigma \,\middle|\, \begin{array}{c} (\beta_i, \alpha_i, \gamma_i) \\ (\beta_i - \alpha_i, \alpha_i, \gamma_i) \end{array} \right] f(z\sigma) d\sigma. \qquad (19)$$

A list of OPEN problems ...