# Канонични параметри на времеподобни повърхнини с паралелен нормиран вектор на средната кривина в $\mathbb{E}^4_1$

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**Lund-Regge problem:**

Find a minimal number of functions, satisfying some natural conditions, that determine the surface up to a motion in Euclidean or pseudo-Euclidean space.

[Lund F., Regge T., *Unified approach to strings and vortices with soliton solutions*. Phys. Rev. D, 14, no. 6 (1976), 1524–1536]

The problem is solved for zero mean curvature surfaces of co-dimension two in the Euclidean 4-space $\mathbb{E}^4$, the Minkoswki space $\mathbb{E}^4_1$ and the pseudo-Euclidean space with neutral metric $\mathbb{E}^4_2$.

# Minimal surfaces in $\mathbb{E}^4$

## Tribuzy and Guadalupe [Rend. Semin. mat. R. Univ. Padova, 1985]

The Gauss curvature $K$ and the curvature of the normal connection $\varkappa$ of any **minimal non-super-conformal surface** parametrized by special isothermal parameters in the Euclidean space $\mathbb{E}^4$ satisfy the following system of partial differential equations

$$(K^2 - \varkappa^2)^{\frac{1}{4}}\,\Delta \ln(K^2 - \varkappa^2) = 8K$$

$$(K^2 - \varkappa^2)^{\frac{1}{4}}\,\Delta \ln \frac{K - \varkappa}{K + \varkappa} = -4\varkappa$$

where $\Delta$ is the Laplace operator $\Delta = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$.

Conversely, any solution $(K, \varkappa)$ to this system determines a unique (up to a rigid motion in $\mathbb{E}^4$) minimal non-super-conformal surface with Gauss curvature $K$ and normal curvature $\varkappa$.

**Remark:**

The class of **minimal super-conformal surfaces** in $\mathbb{E}^4$ is locally equivalent to the class of **holomorphic curves** in $\mathbb{C}^2 \equiv \mathbb{E}^4$.

[Eisenhart L., *Minimal surfaces in Euclidean four-space*, Amer. J. Math. **34** (1912), 215–236]

# Minimal surfaces in $\mathbb{E}^4_1$

The same problem was solved for surfaces with zero mean curvature in $\mathbb{E}^4_1$.

## Alías and Palmer [Math. Proc. Cambridge Philos. Soc., 1998]

**Spacelike surfaces with zero mean curvature** in $\mathbb{E}^4_1$ are described by the following system of partial differential equations

$$(K^2 + \varkappa^2)^{\frac{1}{4}}\,\Delta \ln(K^2 + \varkappa^2) = 8K$$
$$(K^2 + \varkappa^2)^{\frac{1}{4}}\,\Delta \arctan \frac{\varkappa}{K} = 2\varkappa$$

where $K$ and $\varkappa$ are the Gauss curvature and the normal curvature, respectively.

Conversely, any solution $(K, \varkappa)$ to this system determines a unique (up to a rigid motion in $\mathbb{E}^4_1$) spacelike surface with zero mean curvature whose Gauss curvature and normal curvature are the functions $K$ and $\varkappa$, respectively.

# Minimal surfaces in $\mathbb{E}^4_1$

## G. Ganchev, V.M. [Israel J. Math., 2013]

The Gauss curvature $K$ and the normal curvature $\varkappa$ of any **timelike surface with zero mean curvature** satisfy the following system of natural partial differential equations

$$(K^2+\varkappa^2)^{\frac{1}{4}}\,\Delta^h \ln(K^2+\varkappa^2) = 8K$$

$$(K^2+\varkappa^2)^{\frac{1}{4}}\,\Delta^h \arctan\frac{\varkappa}{K} = 2\varkappa$$

where $\Delta^h$ denotes the hyperbolic Laplace operator $\Delta^h = \frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2}$.

Conversely, any solution $(K,\ \varkappa)$ to the above system, determines a unique (up to a rigid motion in $\mathbb{E}^4_1$) timelike surface with zero mean curvature such that $K$ is the Gauss curvature and $\varkappa$ is the normal curvature of the surface.

# Minimal surfaces in $\mathbb{E}^4_2$

**G. Ganchev, K. Kanchev [Comptes rendus de l'Académie bulgare des Sciences, 2019]**

**Spacelike surfaces with zero mean curvature** (maximal spacelike surfaces) in $\mathbb{E}^4_2$ are characterized by the following system of partial differential equations:

$$\begin{aligned} (K^2 - \varkappa^2)^{\frac{1}{4}}\, \Delta \ln(K^2 - \varkappa^2) &= 8K \\ (K^2 - \varkappa^2)^{\frac{1}{4}}\, \Delta \ln \frac{K - \varkappa}{K + \varkappa} &= -4\varkappa \end{aligned} \qquad K^2 - \varkappa^2 > 0.$$

The Gauss curvature $K$ and the normal curvature $\varkappa$ of any **maximal spacelike surface** in $\mathbb{E}^4_2$ satisfy the condition

$$K^2 - \varkappa^2 \geq 0.$$

The equality case is the analogue of the super-conformal minimal surfaces in the Euclidean space $\mathbb{E}^4$.

# Minimal surfaces in $\mathbb{E}^4_2$

## Y. Aleksieva, V.M. [J. Geom. Phys., 2019]

The Gauss curvature $K$ and the normal curvature $\varkappa$ (expressed in terms of the canonical isothermal coordinates) of any **minimal Lorentz surfaces** of general type satisfy the following system of natural partial differential equations:

$$\begin{array}{l} \left|K^2 - \varkappa^2\right|^{\frac{1}{4}} \Delta^h \ln \left|K^2 - \varkappa^2\right| = 8K \\ \left|K^2 - \varkappa^2\right|^{\frac{1}{4}} \Delta^h \ln \left|\dfrac{K + \varkappa}{K - \varkappa}\right| = 4\varkappa \end{array} \qquad K^2 - \varkappa^2 \neq 0. \qquad (1)$$

Conversely, any solution $(K, \varkappa)$ to this system determines a unique (up to a rigid motion in $\mathbb{E}^4_2$) minimal Lorentz surface of general type with Gauss curvature $K$ and normal curvature $\varkappa$ and such that the given parameters are canonical.

### Question 1

*How to introduce canonical parameters and obtain natural equations for other classes of surfaces, different from the minimal ones, in 4-dimensional spaces?*

### Question 2

*Can we solve the Lund-Regge problem for other classes of surfaces, different from the minimal ones, in 4-dimensional spaces?*

We solve this problem for **surfaces with parallel normalized mean curvature vector field** in pseudo-Euclidean 4-spaces.

**Classification results on surfaces with parallel mean curvature vector field:**

- Surfaces with parallel mean curvature vector field in Riemannian space forms were classified in [B.-Y. Chen, *Geometry of submanifolds*, 1973] and Yau [S. Yau, *Amer. J. Math.*, 1974].
- Spacelike surfaces with parallel mean curvature vector field in pseudo-Euclidean spaces with arbitrary codimension were classified in [B.-Y. Chen, *J. Math. Phys.* 2009] and [B.-Y. Chen, *Cent. Eur. J. Math.*, 2009].
- Lorentz surfaces with parallel mean curvature vector field in arbitrary pseudo-Euclidean space $\mathbb{E}^m_s$ are studied in [B.-Y. Chen, *Kyushu J. Math.*, 2010] and [Y. Fu, Z.-H. Hou, *J. Math. Anal. Appl.*, 2010].
- A survey on submanifolds with parallel mean curvature vector in Riemannian manifolds as well as in pseudo-Riemannian manifolds is presented in [B.-Y. Chen, *Arab J. Math. Sci.*, 2010].

# Introduction

## Definition

A submanifold in a Riemannian manifold is said to have ***parallel normalized mean curvature vector field*** if the mean curvature vector is non-zero and the unit vector in the direction of the mean curvature vector is parallel in the normal bundle [B.-Y. Chen, *Monatsh. Math.*, 1980].

- Every analytic surface with parallel normalized mean curvature vector in the Euclidean $m$-space $\mathbb{R}^m$ must either lie in a 4-dimensional space $\mathbb{R}^4$ or in a hypersphere of $\mathbb{R}^m$ as a minimal surface [B.-Y. Chen, *Monatshefte für Mathematik* 1980].
- Spacelike submanifolds with parallel normalized mean curvature vector field in a general de Sitter space are studied in [Shu, S., *J. Math. Phys. Anal. Geom.*, 2011)].

We study **timelike surfaces with parallel normalized mean curvature vector field in the Minkowski 4-space $\mathbb{E}^4_1$.**

**Our aim:** To describe these surfaces in terms of minimal number of functions satisfying a minimal number of partial differential equations.

We prove that the surfaces with PNMCVF can be described in terms of three functions satisfying a system of three partial differential equations.

**Our approach:** To introduce special geometric parameters on each such surface (***canonical parameters***).

# Canonical parameters on timelike surfaces with PNMCVF

$\mathbb{R}^4_1$ – the four-dimensional Minkowski space with the metric $\langle ., . \rangle$ of signature $(3, 1)$.

Let $\mathcal{M} : z = z(u, v),\ (u, v) \in \mathcal{D},\ (\mathcal{D} \subset \mathbb{R}^2)$ be a local parametrization on a timelike surface free of minimal points.

$\widetilde{\nabla}$ and $\nabla$ - the Levi Civita connections on $\mathbb{R}^4_1$ and $\mathcal{M}$, respectively. The formulas of Gauss and Weingarten:

$$\widetilde{\nabla}_x y = \nabla_x y + \sigma(x, y);$$
$$\widetilde{\nabla}_x \xi = -A_\xi x + D_x \xi,$$

The mean curvature vector field $H = \dfrac{1}{2} \operatorname{tr} \sigma$.

# Canonical parameters on timelike surfaces with PNMCVF

Locally, there exist parameters $(u, v)$ on a timelike surface $\mathcal{M}$, such that:

$$g = -f^2(u,v)(du \otimes dv + dv \otimes du), \quad f(u,v) > 0.$$

The coefficients of the first fundamental form are:

$$E = \langle z_u, z_u \rangle = 0; \quad F = \langle z_u, z_v \rangle = -f^2(u,v); \quad G = \langle z_v, z_v \rangle = 0.$$

We consider the pseudo-orthonormal tangent frame field: $x = \dfrac{z_u}{f}, \; y = \dfrac{z_v}{f}$.

$$\langle x, x \rangle = 0, \; \langle x, y \rangle = -1, \; \langle y, y \rangle = 0.$$

The mean curvature vector field is given by:

$$H = -\sigma(x, y).$$

# Canonical parameters on timelike surfaces with PNMCVF

Let $H \neq 0$ and $\{x, y, n_1, n_2\}$ – pseudo-orthonormal frame field such that $H = \nu n_1$, $\nu = ||H||$, $n_2$ is determined up to orientation.

**Remark**: The frame field $\{x, y, n_1, n_2\}$ is geometrically determined:

$x, y$ are the two lightlike directions in the tangent space;

$n_1$ is the unit normal vector field collinear with the mean curvature vector field $H$;

$n_2$ is determined by the condition that $\{n_1, n_2\}$ is an orthonormal frame field of the normal bundle ($n_2$ is determined up to a sign).

We call this pseudo-orthonormal frame field $\{x, y, n_1, n_2\}$ a *geometric frame field* of the surface.

# Canonical parameters on timelike surfaces with PNMCVF

Derivative formulas:

$$
\begin{array}{ll}
\widetilde{\nabla}_x x = \gamma_1 x + \lambda_1 n_1 + \mu_1 n_2 & \widetilde{\nabla}_x n_1 = -\nu x + \lambda_1 y + \beta_1 n_2 \\
\widetilde{\nabla}_x y = -\gamma_1 y - \nu n_1 & \widetilde{\nabla}_y n_1 = \lambda_2 x - \nu y + \beta_2 n_2 \\
\widetilde{\nabla}_y x = -\gamma_2 x - \nu n_1 & \widetilde{\nabla}_x n_2 = +\mu_1 y - \beta_1 n_1 \\
\widetilde{\nabla}_y y = \gamma_2 y + \lambda_2 n_1 + \mu_2 n_2 & \widetilde{\nabla}_y n_2 = \mu_2 x - \beta_2 n_1
\end{array}
$$

where $\gamma_1 = \dfrac{f_u}{f^2} = x(\ln f),\ \gamma_2 = \dfrac{f_v}{f^2} = y(\ln f).$

# Canonical parameters on timelike surfaces with PNMCVF

## Proposition

*Let $\mathcal{M}$ be a timelike surface in the Minkowski space $\mathbb{R}^4_1$. Then, $\mathcal{M}$ has* ***parallel mean curvature vector field*** *if and only if $\beta_1 = \beta_2 = 0$ and $\nu = const$.*

## Proposition

*Let $\mathcal{M}$ be a timelike surface in the Minkowski space $\mathbb{R}^4_1$. Then, $\mathcal{M}$ has* ***parallel normalized mean curvature vector field*** *if and only if $\beta_1 = \beta_2 = 0$ and $\nu \neq const$.*

# Canonical parameters on timelike surfaces with PNMCVF

We consider timelike surfaces with parallel normalized mean curvature vector field (PNMCVF), i.e. we assume that $\beta_1 = \beta_2 = 0$ and $\nu \neq const$.

The surfaces with parallel normalized mean curvature vector field can be divided into two main classes:

- $K - H^2 \neq 0$ (which is equivalent to $\mu_1\mu_2 \neq 0$) in a sub-domain;
- $K - H^2 = 0$ (which is equivalent to $\mu_1\mu_2 = 0$) in a sub-domain.

# Surfaces satisfying $K - H^2 \neq 0$

In the case $K - H^2 \neq 0$, i.e. $\mu_1\mu_2 \neq 0$, there exist smooth functions $\varphi(u) > 0$ and $\psi(v) > 0$ such that:

$$f^2|\mu_1| = \varphi(u); \quad f^2|\mu_2| = \psi(v).$$

We consider the following change of the parameters:

$$\overline{u} = \int_{u_0}^{u} \sqrt{\varphi(u)}\, du + \overline{u}_0, \quad \overline{u}_0 = const,$$
$$\overline{v} = \int_{v_0}^{v} \sqrt{\psi(v)}\, dv + \overline{v}_0, \quad \overline{v}_0 = const.$$

The second fundamental tensor $\sigma$:

$$\sigma(\overline{x}, \overline{x}) = \overline{\lambda}\, n_1 + \overline{\mu}\, n_2;$$
$$\sigma(\overline{y}, \overline{y}) = \frac{\varepsilon_2}{\varepsilon_1}\overline{\lambda}\, n_1 + \frac{\varepsilon_2}{\varepsilon_1}\overline{\mu}\, n_2.$$

where $sign(\mu_1) = \varepsilon_1$, $\varepsilon_1 = \pm 1$ and $sign(\mu_2) = \varepsilon_2$, $\varepsilon_2 = \pm 1$.

# Surfaces satisfying $K - H^2 \neq 0$

$(\overline{u}, \overline{v})$ are special isotropic parameters:

$$g = -\overline{f}^2(\overline{u}, \overline{v})(d\overline{u} \otimes d\overline{v} + d\overline{v} \otimes d\overline{u}), \qquad \overline{f}(\overline{u}, \overline{v}) = \frac{1}{\sqrt{|\overline{\mu}|}}.$$

## Definition

Let $\mathcal{M}$ be a timelike surface with parallel normalized mean curvature vector field in $\mathbb{R}^4_1$ and $K - H^2 \neq 0$. The isotropic parameters $(u, v)$ are said to be ***canonical*** if the metric function $f$ is expressed by:

$$f(u, v) = \frac{1}{\sqrt{|\mu|}}, \quad \mu \neq 0.$$

## Proposition

*Each timelike surface with parallel normalized mean curvature vector field satisfying $K - H^2 \neq 0$ locally admits canonical parameters.*

# Surfaces satisfying $K - H^2 \neq 0$

With respect to canonical isotropic parameters the derivative formulas are:

$$\begin{array}{ll}
\widetilde{\nabla}_x x = \gamma_1 x + \lambda n_1 + \mu n_2; & \widetilde{\nabla}_x n_1 = -\nu x + \lambda y; \\
\widetilde{\nabla}_x y = -\gamma_1 y - \nu n_1; & \widetilde{\nabla}_y n_1 = -\varepsilon \lambda x - \nu y; \\
\widetilde{\nabla}_y x = -\gamma_2 x - \nu n_1; & \widetilde{\nabla}_x n_2 = +\mu y; \\
\widetilde{\nabla}_y y = \gamma_2 y - \varepsilon \lambda n_1 - \varepsilon \mu n_2; & \widetilde{\nabla}_y n_2 = -\varepsilon \mu x,
\end{array}$$

where $\varepsilon = 1$ in the case $K - H^2 > 0$, and $\varepsilon = -1$ in the case $K - H^2 < 0$.

Geometric meaning of the canonical parametrization: if $(u, v)$ are canonical isotropic parameters, then $x = \dfrac{z_u}{f}$ and $y = \dfrac{z_v}{f}$ satisfy the relation:

$$\sigma(x, x) = -\sigma(y, y), \quad \text{in the case } K - H^2 > 0;$$
$$\sigma(x, x) = \sigma(y, y), \quad \text{in the case } K - H^2 < 0.$$

Moreover, the functions $\gamma_1$ and $\gamma_2$ are expressed in terms of the function $\mu$ as follows:

$$\gamma_1 = -\frac{|\mu|_u}{2\sqrt{|\mu|}}, \quad \gamma_2 = -\frac{|\mu|_v}{2\sqrt{|\mu|}}. \tag{2}$$

Then, from the integrability conditions we obtain the following system of PDEs:

$$\begin{aligned} &\nu_u + \lambda_v = \lambda(\ln|\mu|)_v; \\ &\lambda_u - \varepsilon\nu_v = \lambda(\ln|\mu|)_u; \\ &|\mu|\,(\ln|\mu|)_{uv} = -\nu^2 - \varepsilon(\lambda^2 + \mu^2). \end{aligned}$$

# Fundamental Theorem

## Fundamental Theorem 1 [V. Bencheva, V.M., 2023]

Let $\lambda(u,v)$, $\mu(u,v)$ and $\nu(u,v)$ be smooth functions, $\mu \neq 0$, $\nu \neq const$, defined in a domain $\mathcal{D}$, $\mathcal{D} \subset \mathbb{R}^2$, and satisfying the conditions

$$\begin{aligned} &\nu_u + \lambda_v = \lambda(\ln|\mu|)_v; \\ &\lambda_u - \varepsilon\nu_v = \lambda(\ln|\mu|)_u; \\ &|\mu|\,(\ln|\mu|)_{uv} = -\nu^2 - \varepsilon(\lambda^2 + \mu^2), \end{aligned} \tag{3}$$

where $\varepsilon = \pm 1$. If $\{x_0, y_0, (n_1)_0, (n_2)_0\}$ is a pseudo-orthonormal frame at a point $p_0 \in \mathbb{R}^4_1$, then there exists a subdomain $\mathcal{D}_0 \subset \mathcal{D}$ and a unique timelike surface $\mathcal{M} : z = z(u,v),\ (u,v) \in \mathcal{D}_0$ with parallel normalized mean curvature vector field, such that $\mathcal{M}$ passes through $p_0$, $\{x_0, y_0, (n_1)_0, (n_2)_0\}$ is the geometric frame of $\mathcal{M}$ at the point $p_0$, the functions $\lambda(u,v)$, $\mu(u,v)$, $\nu(u,v)$ are the geometric functions of the surface, and $K - H^2 > 0$ in the case $\varepsilon = 1$, resp. $K - H^2 < 0$ in the case $\varepsilon = -1$. Furthermore, $(u,v)$ are canonical isotropic parameters of $\mathcal{M}$.

# Surfaces satisfying $K - H^2 = 0$

In the case $K - H^2 = 0$, i.e. $\mu_1 \mu_2 = 0$, $\mu_1^2 + \mu_2^2 \neq 0$:

$$\nu = \nu(u)$$

and we have two PDEs:

$$\nu_u + \lambda_v = \lambda (\ln |\mu|)_v;$$

$$|\mu| (\ln |\mu|)_{uv} = -\nu^2.$$

## Fundamental Theorem 2 [V. Bencheva, V.M., 2023]

Let $\lambda(u,v)$, $\mu(u,v)$ and $\nu(u)$ be smooth functions, $\mu \neq 0$, $\nu \neq const$, defined in a domain $\mathcal{D}$, $\mathcal{D} \subset \mathbb{R}^2$, and satisfying the conditions

$$\begin{aligned} \nu_u + \lambda_v &= \lambda(\ln|\mu|)_v; \\ |\mu|\,(\ln|\mu|)_{uv} &= -\nu^2. \end{aligned} \tag{4}$$

If $\{x_0, y_0, (n_1)_0, (n_2)_0\}$ is a pseudo-orthonormal frame at a point $p_0 \in \mathbb{R}^4_1$, then there exists a subdomain $\mathcal{D}_0 \subset \mathcal{D}$ and a unique timelike surface $\mathcal{M} : z = z(u,v),\ (u,v) \in \mathcal{D}_0$ with parallel normalized mean curvature vector field, such that $\mathcal{M}$ passes through $p_0$, $\{x_0, y_0, (n_1)_0, (n_2)_0\}$ is the geometric frame of $\mathcal{M}$ at the point $p_0$, the functions $\lambda(u,v)$, $\mu(u,v)$, $\nu(u)$ are the geometric functions of the surface, and $K - H^2 = 0$. Furthermore, $(u,v)$ are canonical isotropic parameters of $\mathcal{M}$.

# Canonical parameters on timelike surfaces with PNMCVF

**Remark**: The canonical parameters are determined up to the following changes:

$$\bar{u} = \pm u + c_1, \qquad c_1 = const;$$
$$\bar{v} = \pm v + c_2, \qquad c_2 = const.$$

We have proved similar results for surfaces with parallel normalized mean curvature vector field in the Euclidean space $\mathbb{E}^4$, spacelike surfaces with PNMCVF in the Minkowski space $\mathbb{E}^4_1$, and Lorentz surfaces in $\mathbb{E}^4_2$.

Examples of solutions to the systems of PDEs characterizing surfaces with PNMCVF in $\mathbb{E}^4$, $\mathbb{E}^4_1$, and $\mathbb{E}^4_2$, can be found in the class of the so-called meridian surfaces – 2-dimensional surfaces lying on rotational hypersurfaces in $\mathbb{E}^4$, $\mathbb{E}^4_1$, or $\mathbb{E}^4_2$, resp.

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*Thank you for your attention!*