Ternary self-dual codes, Hadamard matrices and related designs

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Ternary extremal self-dual codes are known for the following lengths $n \equiv 0 \pmod{12}$: $n = 12$: the extended Golay code, unique up to equivalence; $n = 24$: there are exactly two inequivalent codes: the extended quadratic-residue code [1] and the Pless symmetry code $C(11)$ [5], [6]; $n = 36$: only one code is known: the Pless symmetry code $C(17)$ [5], [6]; $n = 48$: two codes are known: the extended quadratic-residue code and the Pless symmetry code $C(23)$; $n = 60$: three codes are known: the extended quadratic-residue code, the Pless symmetry code $C(29)$, and a code found by Nebe and Villar [4].

It is known [6] that the Pless symmetry code $C(q)$ of length $n = 2q + 2$, where $q \equiv -1 \pmod 3$ is an odd prime power, contains a set of n codewords of weight n, which after replacing every entry equal to 2 with -1 form the rows of a Hadamard matrix equivalent to the Paley-Hadamard matrix of type II. In particular, the Pless symmetry code $C(17)$ contains the rows of a Hadamard matrix P of Paley type II, having a full automorphism group of order $4 \cdot 17(17^2 - 1) = 19584$, and the rows of P span the code $C(17)$.

It was shown in [8] that the code $C(17)$ contains a second equivalence class of Hadamard matrices of order 36 having as rows codewords of $C(17)$. Any matrix H from the second equivalence class has a full automorphism group of order 72 and the rows of H span the code $C(17)$. In addition, H is equivalent to a regular Hadamard matrix H' such that the symmetric 2-$(36, 15, 6)$ design D with a $(0, 1)$-incidence matrix A obtained by replacing every entry 1 of H' with 0 and every entry -1 of H' with 1 has a trivial full automorphism group, and the row span of A over $GF(3)$ is equivalent to the Pless symmetry code $C(17)$.

Huffman [3] proved that any extremal ternary self-dual code of length 36 that admits an automorphism of prime order $p > 3$ is monomially equivalent to the Pless symmetry code. More recently, Eisenbarth and Nebe [2] extended Huffman's result by proving that the Pless symmetry code is the unique (up to monomial equivalence) ternary extremal self-dual code of length 36 that admits an automorphism of order 3. In addition, it was proved in [2, Theorem 5.1] that if C is an extremal ternary self-dual code of length 36 then either C is equivalent to the Pless symmetry code or the full automorphism group of C is a subgroup of the cyclic group of order 8.

In this talk, we report on the existence of a regular Hadamard matrix H^* which is monomially equivalent to the Paley-Hadamard matrix of type II such that the symmetric 2-$(36, 15, 6)$ design associated with H^* has a full automorphism group of order 24 and its (0,1)-incidence matrix spans a code equivalent to $C(17)$ [7]. Motivated by this and the results from [2], we classified all symmetric 2-$(36, 15, 6)$ designs that admit an automorphism of order 2 and their incidence matrices span an extremal ternary self-dual code of length 36 [7]. The results of this classification imply the following.

Theorem 1. *(a) Up to isomorphism, there exists exactly one symmetric 2-*$(36, 15, 6)$ *design D that admits an automorphism of order 2 and its incidence matrix spans an extremal ternary self-dual code of length 36.*

(b) The full automorphism group G of D is of order 24, and G is isomorphic to the symmetric group S_4.

(c) The regular Hadamard matrix associated with D is equivalent to the Paley-Hadamard matrix of type II.

(d) The ternary code spanned by the incidence matrix of D is equivalent to the Pless symmetry code.

References

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