Introduction to Resolution of Singularities

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In these lectures I will introduce the very basic objects that help study **resolutions of singularities**, from the point of view of valuations. This is the historic strategy pioneered by Zariski and later by Abhyankar. My goal is to present a proof of **the resolution of surfaces in characteristic zero**, via the **local uniformization** problem. This approach had lost momentum after Hironaka's acclaimed breakthrough, but has regained interest in the 90s as new ideas emerged in the works of Spivakovsky and Teissier.

In the first lectures I will introduce the basics of singularity theory, that concerns regular rings and will go up to the Jacobian criterion, without providing a detailed proof. I will then present the basics of the **resolution of curves** via the normalization process and its link to Dedeking rings. This will allow me to dive into **valuation theory**, where I will present the the tools and basic invariants that help classify valuations in low dimension.

The main breakthrough of valuation theory into singularity theory comes from the quasi-compactness of the **Riemann-Zariski space**. It is the set of valuations centred on the points of a variety and its topological properties gives us a strategy towards the local uniformization (LU) problem.

LU has been achieved by Zariski for any valuation on any variety in zero characteristic. In the positive characteristic, it has only been achieved over curves, surfaces and solids, at the price of numerous efforts. I will solely present the **resolution of singularities of surfaces in zero characteristic**, which has the advantage of presenting some subtleties that arise in valuation theory.

In the final part I shall bring these results together, by showing how to glue back all the local pieces into a resolution of surfaces. The main ingredient here is Zariski's "Main" theorem.

Along the way, I will show how things can be generalised, without delving into too much details. The course is intended towards an undergraduate level.

Prerequisites:

- commutative algebra: Noetherian rings and modules, integral closure, Krull dimension and affine domains.
- algebraic geometry: varieties (projective and affine), morphisms and rational maps and how they relate to the coordinate rings and function fields.

Logistics: Meetings will take place once a week in **hall 403** at IMI-BAS and the lectures will be 90 minutes long on average.