Ranks of Monoids of Endomorphisms, Partial Automorphisms and Injective Partial Endomorphisms of a Finite Undirected Path

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Abstract

In the same way that automorphisms of graphs allow to establish natural connections between Graph Theory and Group Theory, endomorphisms of graphs do the same between Graph Theory and Semigroup Theory. For this reason, it is not surprising that monoids of endomorphisms of graphs have been attracting the attention of several authors over the last decades. In fact, from combinatorial properties to more algebraic concepts have been extensively studied.

Let $G = (V, E)$ be a simple graph (i.e. undirected graph without loops and without multiple edges). Let α be a partial transformation of V. Denote by $\operatorname{Dom}\alpha$ the domain of α and by $\operatorname{Im}\alpha$ the image of α. We say that α is:

- a *partial endomorphism* of G if $\{u, v\} \in E$ implies $\{u\alpha, v\alpha\} \in E$, for all $u, v \in \operatorname{Dom}\alpha$;
- a *weak partial endomorphism* of G if $\{u, v\} \in E$ and $u\alpha \neq v\alpha$ imply $\{u\alpha, v\alpha\} \in E$, for all $u, v \in \operatorname{Dom}\alpha$;
- a *strong endomorphism* of G if $\{u, v\} \in E$ if and only if $\{u\alpha, v\alpha\} \in E$, for all $u, v \in V$;
- a *strong weak endomorphism* of G if $\{u, v\} \in E$ and $u\alpha \neq v\alpha$ if and only if $\{u\alpha, v\alpha\} \in E$, for all $u, v \in V$;
- a *partial automorphism* of G if α is an injective mapping (i.e. a partial permutation) and α and α^{-1} are both partial endomorphisms;
- if α is a full mapping (i.e. $\alpha \in \mathcal{T}(V)$) then to a partial endomorphism (respectively, weak partial endomorphism and partial automorphism) we just call *endomorphism* (respectively, *week endomorphism* and *automorphism*).

Denote by:

- $\operatorname{End}(G)$ the set of all endomorphisms of G;
- $\operatorname{wEnd}(G)$ the set of all weak endomorphisms of G;
- $\operatorname{sEnd}(G)$ the set of all strong endomorphisms of G;
- $\operatorname{swEnd}(G)$ the set of all strong weak endomorphisms of G;
- $\operatorname{Aut}(G)$ the set of all automorphisms of G.
- $\operatorname{wPEnd}(G)$ the set of all weak partial endomorphisms of G;
- $\operatorname{PEnd}(G)$ the set of all partial endomorphisms of G;
- $\operatorname{IEnd}(G)$ the set of all injective partial endomorphisms of G;
- $\operatorname{PAut}(G)$ the set of all partial automorphisms of G;

Clearly, $\mathrm{End}(G)$, $\mathrm{wEnd}(G)$, $\mathrm{sEnd}(G)$, $\mathrm{swEnd}(G)$, $\mathrm{Aut}(G)$, $\mathrm{wPEnd}(G)$, $\mathrm{PEnd}(G)$, $\mathrm{IEnd}(G)$ and $\mathrm{PAut}(G)$ are monoids under composition of maps with the identity mapping id as the identity element. Moreover, $\mathrm{Aut}(G)$ is also a group and $\mathrm{PAut}(G)$ is an inverse semigroup.

The *rank* of a monoid S, denoted by $\operatorname{rank} S$, is the least number of generators of S. We focus our attention on this important notion of Semigroup Theory, which has been, in recent years, the subject of intensive research.

We study the widely considered endomorphisms, weak endomorphisms, partial automorphisms and, more generally, injective partial endomorphisms of a finite undirected path P_n with $n \in \mathbb{N}$ vertices from monoid generators perspective. Our main objective is to give formulas for the ranks of the monoids $\mathrm{wEnd}(P_n)$, $\mathrm{End}(P_n)$, $\mathrm{sEnd}(P_n)$, $\mathrm{swEnd}(P_n)$, $\mathrm{Aut}(P_n)$, $\mathrm{IEnd}(P_n)$ and $\mathrm{PAut}(P_n)$. We also study Green's relations, regularity, and cardinality for some of these monoids.